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TERNARY AND QUATERNARY ELIMINATION.*

By Arthur B. Coble.

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1. Introduction. It is the aim of the present paper to exhibit explicit determinants of covariant character which will, in certain cases, be the resultant R of n forms in n variables, $(\alpha_i y)^{l_i}$ $(i = 1, \cdots, n)$, multiplied by a relatively simple factor also of covariant character.

Van der Waerden [1] (p. 20) defines R to be the G. C. D. of $l+1$ determinants found in the "dialytic array" of forms of order $l+1$ where $l = \Sigma(l_i - 1)$. Macaulay [2] (p. 4) had earlier defined R to be the G. C. D. of all of the determinants of this array and proves (pp. 10-11) that $D = E \cdot R$, where D is a specific determinant of the array and E is a specific minor of D. This result is quite precise but neither D, nor the "extraneous factor" E are invariantive and both are determinants of high order.

The method used is an extension of that initiated by F. Morley [3] and developed more extensively by Morley and Coble [4]. In the first of these papers Morley introduced a covariant J_k which, for the general case above, has the expression,

$$J_k(x^h, y^k) = (\alpha_1 \alpha_2 \cdots \alpha_n) \, \Sigma \prod_{i=1}^{i=n} (\alpha_i x)^{l_i - k_i - 1} (\alpha_i y)^{k_i}, \tag{1}$$

$$h + k = l, \qquad 0 \leq k_i \leq l_i - 1,$$

where Σ is extended over all sets of positive integer values of the k_i for which also

$$k_1 + k_2 + \cdots + k_n = k. \tag{2}$$

In the second paper it is proved that, when the n equations, $(\alpha_i y)^{l_i} = 0$, have a fixed common solution x, then $J(x^h, y^k)$ is in the module determined by the n given forms. Thus, for

$$(\alpha_i x)^{l_i} = 0,$$

$$J(x^h, y^k) \equiv (\lambda_1 y)^{k-l_1} \cdot (\alpha_1 y)^{l_1} + \cdots + (\lambda_n y)^{k-l_n} \cdot (\alpha_n y)^{l_n}. \tag{3}$$

This identity (3) in y together with suitable dialytic equations enabled

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* Received May 18, 1946.

Morley and Coble to obtain determinants $D = R$ in all of the ternary cases covered by

$$l_1 \leq l_2 \leq l_3 \leq l_1 + l_2 + 1. \tag{4}$$

It was used later by T. W. Moore [5] to obtain a variety of forms for a particular binary resultant. Its usefulness in the quaternary domain was much more limited. There only the lowest order l_1 was arbitrary, the others being quite limited.

We propose here to use this identity more widely and to cover all cases of ternary and quaternary elimination. The new determinants D obtained are all of the form ER but the extraneous factor E is itself a covariant of the given forms, usually of very simple character.

In the course of the work we use constantly the number of coefficients in a given ternary or quaternary form. This number, a binomial coefficient, will be indicated by the more convenient notation defined by

$$[n]_k = \binom{n}{k} = {}_nC_k.$$

2. The Ternary Identities. In order to avoid subscripts we shall use forms,

$$(\alpha y)^m, (\beta y)^n, (\gamma y)^p \quad [m \leq n \leq p \geq m + n + 2]. \tag{5}$$

The fundamental identity (3) then reads:

$$\text{(I)} \quad J(x^h, y^k) + (\lambda y)^{k-m} \cdot (\alpha y)^m + (\mu y)^{k-n} \cdot (\beta y)^n + (\nu y)^{k-p} \cdot (\gamma y)^p \equiv 0 \text{ (in}$$

$$h + k = l = m + n + p - 3.$$

Naturally, if any of the exponents $k - m$, $k - n$, $k - p$ are negative, the corresponding terms do not appear. We shall assume merely that

$$h < n + p, \qquad k < n + p.$$

Then, in (I), the coefficients $(\lambda y)^{k-m}$, $(\mu y)^{k-n}$, $(\nu y)^{k-p}$ are no longer necessarily unique since terms of the form $(\kappa_{12} y)^{k-m-n} \cdot (\alpha y)^m \cdot (\beta y)^n$ can appear in either λ or μ, and terms of the form $(\kappa_{13} y)^{k-m-p} \cdot (\alpha y)^m \cdot (\gamma y)^p$ can appear in either λ or ν. When $k \leq m + n$, or when $k \leq m + p$ we restore the uniqueness of the coefficients λ, μ, ν by the respective requirements:

$$\text{(IIa)} \quad (\mu y)^{k-m-n} (\mu z)^m \equiv 0 \text{ (in } y), (\alpha z)^m \not\equiv 0;$$

$$\text{(IIb)} \quad (\nu y)^{k-m-p} (\nu z)^m \equiv 0 \text{ (in } y), (\alpha z)^m \not\equiv 0$$

where z is any arbitrarily chosen point not on $(\alpha y)^m \neq 0$. Indeed, if λ, μ, ν are admissible multipliers and if $k \leq m + n$, $k < n + p$, the multiplier μ may be replaced by $(\mu y)^{k-n} - (\kappa_{12} y)^{k-n-m} \cdot (\alpha y)^m$, where $(\kappa_{12} y)^{k-n-m}$ is an arbitrary form. The requirement (IIa) imposes $[k - n - m + 2]_2$ linear non-homogeneous conditions on the $[k - n - m + 2]_2$ coefficients of $(\kappa_{12} y)^{k-n-m}$. The determinant of the system has elements which are linear in the coefficients of the form $(\alpha y)^m$ and of the m-th order in z. Since the determinant is obviously a covariant it must be a numerical multiple of $[(\alpha z)^m]^{[k-m-n+2]_2}$. Thus, if $(\alpha z)^m \neq 0$, $(\mu y)^{k-n}$ is uniquely defined by (IIa), and similarly $(\nu y)^{k-p}$ is uniquely defined by (IIb), whence also $(\lambda y)^{k-m}$ is uniquely determined.

To the above two types of identities we add a third type derived from equations of the dialytic array. These equations can be expressed in the form,

$$(6) \qquad (\rho x)^{h-m} \cdot (\alpha x)^m \equiv 0 \ (\text{in } \rho), \ (\sigma x)^{h-n} \cdot (\beta x)^n \equiv 0 \ (\text{in } \sigma),$$
$$(\tau x)^{h-p} \cdot (\gamma x)^p \equiv 0 \ (\text{in } \tau).$$

If however $h \geq m + n$ or $h \geq m + p$, the equations obtained from the identities in σ or τ will overlap with those obtained from the identity in ρ. This overlapping may be avoided by the requirement that σ, τ have at z multiple points of orders $h - m - n - 1$, $h - m - p - 1$ respectively, i. e., that

$$(7) \qquad (\sigma y)^{h-m-n} (\sigma z)^m \equiv 0 \ (\text{in } y), \ (\tau y)^{h-m-p} (\tau z)^m \equiv 0 \ (\text{in } y).$$

This requirement defines σ, for example, in the linear system $(\sigma y)^{h-n} + (\rho_{12} y)^{h-m-n} \cdot (\alpha y)^m$, and will bar the overlapping part, $(\rho_{12} y)^{h-m-n} \cdot (\alpha y)^m$, if $(\alpha z)^m \neq 0$. We can then replace the second identity in (6) by

$$(\sigma x)^{h-n} \cdot (\beta x)^n + b (\sigma x)^{h-m-n} (\sigma z)^m \equiv 0 \ (\text{in } \sigma),$$

since the first term vanishes identically due to $(\beta x)^n = 0$, and the second term vanishes due to (7). The constant b is introduced in order to cover the lack of homogeneity in the coordinates of the fixed common point x, and in the coordinates of the extraneous point. Naturally in using such an identity we must eliminate both the combinations of x to the degree h and b times the combinations of x to the degree $h - m - n$. Finally we replace the coefficients σ of the form $(\sigma y)^{h-n}$ by the $(h - n)$-th combinations of a linear form (ηy) to obtain the third type of identity:

$$(\text{IIIa}) \qquad (\eta x)^{h-m} \cdot (\alpha x)^m \equiv 0 \ (\text{in } \eta);$$

(IIIb) $$(\eta x)^{h-n}\cdot(\beta x)^n + b(\eta x)^{h-m-n}(\eta z)^m \equiv 0 \text{ (in } \eta);$$

(IIIc) $$(\eta x)^{h-p}\cdot(\gamma x)^p + c(\eta x)^{h-m-p}(\eta z)^m \equiv 0 \text{ (in } \eta).$$

In case the terms in b or c disappear in the identities (IIIb) or (IIIc), due to $h < m+n$ or $h < m+p$, we shall refer to them as (IIIb′) or (IIIc′) respectively.

3. Determinants D for given m, n, p. With a given set of orders m, n, p restricted as in **2** (5) we have a variety of cases depending on the choice of h, k in

(8) $$h + k = l = m + n + p - 3.$$

Each case yields a determinant D which contains the factor R. We distinguish these cases by the value of S in

(9) $$h = m + S - 3, \qquad k = n + p - S$$
$$(S = 1, 2, \cdots, p + n - m + 2),$$

with the exception that, if $m = 1$, the first and last values of S are to be omitted. On the value of k there will depend the occurrence of all, part, or none, of λ, μ, ν in the identity (I) and the use of the identities (IIa), (IIb). On the value of h there will depend the need for identities of the simple type (IIIa), (IIIb′), (IIIc′), or the replacement of (IIIb′), (IIIc′) by (IIIb), (IIIc). In all of the cases our object is to obtain, from a complete use of the identities involved, just enough equations to eliminate the combinations x^h, and such of the coefficients λ, μ, ν and combinations bx^{h-m-n}, cx^{h-m-p} as occur, by using the determinant D of the equations.

The range of values of S may be divided into sets, each set contributing a particular form of D. This division is as follows:

	S	k	h	
Set I 1:	1	$n+p-1$	$m-2$	; $\lambda, \mu(\kappa_{12}), \nu(\kappa$
	2	$n+p-2$	$m-1$	;
Set I 2:	3	$n+p-3$	m	; $\lambda, \mu(\kappa_{12}), \nu(\kappa$
. . . . .				
	$n-m$	$p+m$	$n-3$	;
Set II 1:	$n-m+1$	$p+m-1$	$n-2$	; $\lambda, \mu(\kappa_{12}), \nu, \alpha$
	$n-m+2$	$p+m-2$	$n-1$	;
Set II 2:	$n-m+3$	$p+m-3$	n	; $\lambda, \mu(\kappa_{12}), \nu, \alpha$
. . . . .				

	n	p	$m+n-3$	;
Set III 1:	$n+1$	$p-1$	$m+n-2$	; $\lambda, \mu(\kappa_{12}), \alpha, \beta$
	$n+2$	$p-2$	$m+n-1$	;
Set III 2:	$n+3$	$p-3$	$m+n$	; $\lambda, \mu(\kappa_{12}), \alpha, \beta(b)$
. . . . .				
	$p-m$	$m+n$	$p-3$	;
Set IV 1:	$p-m+1$	$m+n-1$	$p-2$	; $\lambda, \mu, \alpha, \beta(b)$
	$p-m+2$	$m+n-2$	$p-1$	;
Set IV 2:	$p-m+3$	$m+n-3$	p	; $\lambda, \mu, \alpha, \beta(b), \gamma$
. . . . .				
	p	n	$p+m-3$	;
Set V 1:	$p+1$	$n-1$	$p+m-2$	; $\lambda, \alpha, \beta(b), \gamma$
	$p+2$	$n-2$	$p+m-1$	;
Set V 2:	$p+3$	$n-3$	$p+m$	; $\lambda, \alpha, \beta(b), \gamma(c)$
. . . . .				
	$p+n-m$	m	$p+n-3$	;
Set VI 1:	$p+n-m+1$	$m-1$	$p+n-2$	; $\alpha, \beta(b), \gamma(c)$
	$p+n-m+2$	$m-2$	$p+m-1$.	

The quantities at the end of a particular set need some explanation. The occurrence of μ indicates that the coefficients μ can be eliminated by using the identity (I) alone, that of $\mu(\kappa_{12})$ indicates that identities (I) and (IIa) must both be used to eliminate the coefficients μ. The occurrence of β indicates that the dialytic identity (IIIb′) is used whereas that of $\beta(b)$ indicates that (IIIb) must be employed. Thus a glance down the last column shows that the expression for D is different in each set. However the main differences appear as we pass through Sets I 2, II 2, III 2, IV 2, V 2 and we indicate below the determinants D for these five cases. A trifling omission in any one of these five will cover either the preceding or the following set in each of which S has only two values.

The determinant D for Set I 2 has the following expression and value:

Set I 2: $D = R \cdot [(\alpha z)^m]^d =$

	$[m+S-1]_2 x^h$	$[n+p-m-S+2]_2 \lambda$	$[p-S+2]_2 \mu$	$[n-S+2]_2 \nu$
(I): $[n+p-S+2]_2$	$\alpha\beta\gamma$	α	β	γ
(IIa): $[p-S-m+2]_2$	0	0	z^m	0
(IIb): $[n-S-m+2]_2$	0	0	0	z^m
(IIIa): $[S-1]_2$	α	0	0	0

$(d = [p-S-m+2]_2 + [n-S-m+2]_2)$.

The binomial coefficients at the left of the array give the number of equations obtained from the identities indicated; those at the top of the array give the number of terms eliminated. For the blocks of elements within the array, the degrees in the coefficients of the given forms and in z are indicated. The verification that the sum of the rows of D is equal to the sum of the columns will not be reproduced here. The expansion of a determinant of the form of D is simple, and it is easily verified that the degrees of D in the coefficients of the given forms and z are those of the product given above. That D actually is this product is proved in **6**.

The above D will adequately cover the Set I 1. For, if S is 1 or 2, $[S-1]_2 = 0$, and the identity (IIIa) will be missing. Thus the verifications mentioned above need not be repeated. It may be used to cover the Set II 1 also. For, if S is $n-m+1$ or $n-m+2$, $[n-S-m+2]_2 = 0$, and the identity (II b) will not appear.

With these explanations we may, without further ado, set down the typical forms of the determinants D for the Sets II 2, III 2, IV 2, V 2, for each of which the two verifications mentioned above are easily made, and for each of which simple alterations cover the adjacent cases.

Set II 2: $D = R \cdot [(\alpha z)^m]^d =$

	$[m+S-1]_2 x^h$	$[n+p-m-S+2]_2 \lambda$	$[p-S+2]_2 \mu$	$[n-S+2]_2 \nu$
(I): $[n+p-S-2]_2$	$\alpha\beta\gamma$	α	β	γ
(IIa): $[p-m-S+2]_2$	0	0	z^m	0
(IIIa): $[S-1]_2$	α	0	0	0
(IIIb′): $[m-n+S-1]_2$	β	0	0	0

$(d = [p-m-S+2]_2)$.

Set III 2: $D = R \cdot [(\alpha z)^m]^d =$

	$[m+S-1]_2 x^h$	$[n+p-m-S+2]_2 \lambda$	$[p-S+2]_2 \mu$	$[S-n-1]_2 \nu x^{S-n-1}$
(I): $[n+p-S+2]_2$	$\alpha\beta\gamma$	α	β	0
(IIa): $[p-m-S+2]_2$	0	0	z^m	0
(IIIa): $[S-1]_2$	α	0	0	0
(IIIb): $[m-n+S-1]_2$	β	0	0	z^m

$(d = [S-n-1]_2 + [p-m-S+2]_2)$.

Set IV 2: $D = R \cdot [(\alpha z)^m]^d =$

	$[m+S-1]_2 x^h$	$[n+p-m-S+2]_2 \lambda$	$[p-S+2]_2 \mu$	$[S-n-1]_2 b x^{S-n-3}$
(I) : $[n+p-S+2]_2$	$\alpha\beta\gamma$	α	β	0
(IIIa) : $[S-1]_2$	α	0	0	0
(IIIb) : $[m-n+S-1]_2$	β	0	0	z^m
(IIIc′) : $[m+S-p-1]_2$	γ	0	0	0

$(d = [S-n-1]_2)$.

Set V 2: $D = R \cdot [(\alpha z)^m]^d =$

	$[m+S-1]_2 x^h$	$[n+p-m-S+2]_2 \lambda$	$[S-n-1]_2 b x^{S-n-3}$	$[S-p-1]_2 c x^{S-n-3}$
(I) : $[n+p-S+2]_2$	$\alpha\beta\gamma$	α	0	0
(IIIa) : $[S-1]_2$	α	0	0	0
(IIIb) : $[m-n+S-1]_2$	β	0	z^m	0
(IIIc) : $[m-p+S-1]_2$	γ	0	0	z^m

$(d = [S-n-1]_2 + [S-p-1]_2)$.

4. Reciprocity among the determinants D for given m, n, p. In all of the above determinants D, the identity (I) involving $J(x^h, y^k)$ has been employed. We consider now the relation between the determinant D for $J(x^h, y^k)$ and the determinant D' for $J(x^k, y^h)$. They arise respectively from values S and S' for which

$$S + S' = p + n - m + 3. \tag{10}$$

Thus the Set III 2 is self-reciprocal, the first member of the set being paired with the last, etc. The Sets III 1 and IV 1 are reciprocal to each other, as also are the pairs II 2, IV 2; II 1, V 1; I 2, V 2; and I 1, VI 1. It is clear that in $J(x^h, y^k)$ and $J(x^k, y^h)$ the matrices $\alpha\beta\gamma$ are transposed. It is also clear from the last column of the table in **3** that, when h and k are interchanged, the terms in λ, μ, ν are interchanged with those in α, β, γ and that $\mu(\kappa_{12})$, $\nu(\kappa_{13})$ are interchanged with $\beta(b)$, $\gamma(c)$ respectively. From the character of the identities involved we may conclude that

(11) *If S, S' satisfy the relation* (10), *then D, D' formed for these values are conjugate determinants.*

From this there follows immediately that

(12) *The number of essentially different determinants D is*
$$(p+n-m+3)/2 \text{ or } (p+n-m+4)/2.$$

5. The non-vanishing of the determinants D. It is essential for our conclusion that D be not identically zero. This is verified by the non-vanishing of D in the particular case

$$(13) \qquad (\alpha y)^m = y_1^m,\ (\beta y)^n = y_2^n,\ (\gamma y)^p = y_3^p;\ z_1 : z_2 : z_3 = 1 : 0 : 0,$$

in which case $(\alpha z)^m = 1$. We carry out the verification for only one specimen case in Set III 1, say the case

$$S = n+2, \qquad h = m+n-1, \qquad k = p-2.$$

Then

$$(14) \qquad J(x^{m+n-1}y^{p-2}) = \Sigma x_1^{i_1} x_2^{i_2} x_3^{i_3} y_1^{m-1-i_1} y_2^{n-1-i_2} y_3^{p-1-i_3}$$
$$(i_1+i_2+i_3 = m+n-1,\ i_1 \leq m-1,\ i_2 \leq n-1).$$

The multiple-point condition on μ is now $(\mu y)^{p-m-n-2}\mu_1^m \equiv 0$. This merely asserts that $[p-m-n]_2$ coefficients of $(\mu y)^{p-n-2}$ are zero and this condition may be dropped if only the remaining coefficients of μ are retained. Thus D takes the relatively simple form

$$(15) \qquad D = \begin{array}{c|ccc} & x_1^{i_1}x_2^{i_2}x_3^{i_3} & \lambda_1^{j_1}\lambda_2^{j_2}\lambda_3^{j_3} & \mu_1^{k_1}\mu_2^{k_2}\mu_3^{k_3} \\ \hline [p]_2 & \alpha\beta\gamma & \alpha & \beta \\ [n+1]_2 & \alpha & 0 & 0 \\ [m+1]_2 & \beta & 0 & 0 \end{array}.$$

The conditions on i_1, i_2, i_3 are given in (14); those on j_1, j_2, j_3 and k_1, k_2, k_3 are

$$(16) \quad j_1+j_2+j_3 = p-m-2;\ k_1+k_2+k_3 = p-n-2,\ k_1 < m.$$

We wish to show that, for the simplified forms (13), every row and column of D has one element 1 with all other elements 0. Thus $D = \pm 1$ and, in general, $D \neq 0$.

We prove this first for the rows. The first set of $[p]_2$ equations, obtained from the identity (I), arises from combinations $y_1^{l_1}y_2^{l_2}y_3^{l_3}$ with $l_1+l_2+l_3 = p-2$. These combinations divide into three aggregates:

(a) $y_1^{m-1-i_1}y_2^{n-1-i_2}y_3^{p-1-i_3}$ found in $J(x^{m+n-1}y^{p-2})$;

(b) $y_1^{j_1+m}y_2^{j_2}y_3^{j_3}$ found in $(\lambda y)^{p-2-m}\cdot y_1^m$;

(c) $y_1^{k_1}y_2^{k_2+n}y_3^{k_3}$ found in $(\mu y)^{p-2-n}\cdot y_2^n$.

A glance at the conditions in (14) and (16) enables us to draw at once the following conclusions. If $l_1 < m$ and $l_2 < n$, the combination is found once and only once in (a). If $l_1 < m$ and $l_2 \geq n$, the combination is found once and only once in (c). If $l_1 \geq m$ the combination is found once and only once in (b). Thus there is only one term in each of these $[p]_2$ equations. Evidently the other equations, obtained by multiplying either x_1^m or x_2^n by proper combinations of x, also contain only one term. Turning to the columns, it is clear that $x_1^{i_1}x_2^{i_2}x_3^{i_3}$ with the restrictions (16), will appear once and only once, with a coefficient α if $i_1 \geq m$, with a coefficient β if $i_2 \geq n$, whereas it will appear in J if both $i_1 < m$ and $i_2 < n$. Similarly, a coefficient λ, or a non-zero coefficient μ, will occur with one and only one combination of the y's so that the last two sets of columns have only one non-zero element.

With similar verifications for the other values of S we may assert that

(17) *The determinants D in* **3** *do not vanish identically.*

6. Determination of the resultant R. The identities which furnish the equations whose determinant is D all are expressed in terms of the given forms and of z. Thus D itself is a non-vanishing covariant. These equations are consistent if x is a common solution of the given forms whence D must contain the irreducible invariant R as a factor. The remaining covariant factor has a degree d in the coefficients of $(\alpha y)^m$ and an order md in the coordinates z. It must therefore be the d-th power of $(\alpha z)^m$. Hence

(18) *The determinants D listed in* **3** *are products of R and the powers of $(\alpha z)^m$ indicated.*

For sufficiently generic $(\alpha y)^m$, say

$$(19) \qquad (\alpha y)^m = A y_1^m + \cdots \qquad (A \neq 0)$$

and the choice of $z = 1, 0, 0 = E_1$ so that $(\alpha z)^m = A$, these determinants D can be materially simplified. We have employed this simplification in part in the particular case treated in **5**. More generally we use the identity (I) with the additional requirement that $(\mu y)^{k-n} = 0$ and $(\nu y)^{k-p} = 0$ have multiple points of orders $k - n - m + 1$ and $k - p - m + 1$ respectively at the reference point E_1. Thus the identities (IIa), (IIb) are eliminated by the fact that $(\mu y)^{k-n}$ and $(\nu y)^{k-p}$ have a smaller number of coefficients which must be eliminated. Similarly, we use the identities (6) with the additional requirement that $(\sigma y)^{h-n} = 0$ and $(\tau y)^{h-p} = 0$ have multiple points of orders $h - n - m + 1$ and $h - p - m + 1$ respectively at E_1. Then the equations

of the dialytic array obtained from the zero coefficients of σ and τ are dropped, and the remaining equations are linearly independent. Hence

(20) *If* $(\alpha y)^m = Ay_1^m + \cdots$ *with* $A \neq 0$, *and if the coefficients of the terms of degree greater than* $m-1$ *in* y_1 *of the forms* $(\mu y)^{k-n}$, $(\nu y)^{k-p}$, $(\sigma y)^{h-n}$, $(\tau y)^{h-p}$ *are all zero, then the elimination of the combinations* x^h, *and of* λ, μ, ν *from the equations furnished by the identity* (I), *and the additional dialytic equations furnished by* (6), *yields a determinant* D' *which is* $R \cdot A^d$ *where* d *is the power indicated in the table of Section* **3**.

It may be of interest to compare this result with more conventional methods. Let $m, n, p = 2, 3, 7$ and thus $l = 9$. The best results are obtained from median values of S. We use therefore the identity (I) for $J(x^4y^5)$, dropping the coefficient μ_1 of y_1^2 in $(\mu y)^2$. We obtain a determinant D' of order 30 which is $R \cdot A$. On the other hand an extreme value of S as in $J(x^0, y^9)$ yields a determinant of order 55 which is $R \cdot A^{16}$. These are to be compared with Van der Waerden's R as the G. C. D. of three determinants of order 66, or Macaulay's $R = D/D'$ where D, D' are determinants of order 66 and 25 respectively. Obviously the use of J improves very considerably both the order of D and the nature of the factor extraneous to R in D.

7. The unique apolar form. The three forms $(ay)^m$, $(\beta y)^n$, $(\gamma y)^p$, if sufficiently generic, and if in particular $R \neq 0$, have a unique apolar form of class l, $(a\eta)^l$, for which

$$(21) \qquad (\alpha a)^m (a\eta)^{l-m} \equiv 0, \quad (\beta a)^n (a\eta)^{l-n} \equiv 0, \quad (\gamma a)^p (a\eta)^{l-p} \equiv 0.$$

This apolar form can be obtained by bordering properly the determinants D of the table in **3**, or the determinants D' contemplated in (20). We first write the identity (I) involving $J(x^h, y^k)$ with an additional term $c(\eta y)^k$, and then add the equation, $(\xi x)^h = 0$. When $R = 0$, and there is a common point x, the new equations are all satisfied by arbitrary values η (when $c = 0$), provided only that ξ is on the common point x. Thus the determinant B or B', which arises from D or D' respectively by adding a column of η^k's and a row of ξ^h's to border the original determinant with respect to the matrix of its elements which arises from $J(x^h, y^k)$, contains a factor $(\xi x)^h$ when $R = 0$. If we take the reciprocal determinant [cf. (11)] formed for $J(x^k, y^h)$ and border it similarly, it contains a factor $(\xi x)^k$ when $R = 0$. But these reciprocal determinants are conjugate. Hence if either is bordered with both a column and a row of ξ's it will be a multiple of $(\xi x)^{h+k}$ when $R = 0$. But if x is a common point,

$(\xi x)^{h+k}$ is the unique apolar form. Since B or B' is of lower degree than the irreducible invariant R, it must also be the unique apolar form when $R \neq 0$. The presence of a factor $[(\alpha z)^m]^d$ in B or A^d in B' will not alter the argument so that we may state the theorem:

(22) *If the determinants D of the table in* **3**, *or the determinants D' of* (20), *are bordered with respect to the elements in the matrix which arises from $J(x^h, y^k)$ by a column of combinations η^k and a row of combinations η^h, the bordered determinants are expressions for the unique apolar form, $(a\eta)^l$, to within the same factor as appears in the original determinant extraneous to R.*

Thus the unique apolar form may be described as the *evectant* of R with respect to the coefficients of J in the determinant forms of R. Since the converse of the evectant process is the apolarity process we may assert that

(23) *The resultant R of the three given ternary forms is the apolarity invariant of the form $(a\eta)^l$ apolar to the given forms and of the jacobian $(\alpha\beta\gamma)(\alpha y)^{m-1}(\beta y)^{n-1}(\gamma y)^{p-1}$ of the given forms.*

This unique apolar form of degrees $np-1$, $mp-1$, $mn-1$ in the coefficients of the three given forms respectively is discussed much more completely in the following memoir.

The above account, taken in conjunction with (4), covers all cases of ternary elimination. We proceed to cover also all cases of quaternary elimination though no attempt will be made to give more than one determinant in a particular case.

8. Quaternary elimination with $D = R$. With four given quaternary forms, $(\alpha y)^m$, $(\beta y)^n$, $(\gamma y)^p$, $(\delta y)^q$ and

$$m \leq n \leq p \leq q, \quad l = m + n + p + q - 4, \tag{24}$$

Morley and Coble [4] used the identity,

$$\text{(I)} \qquad J(x^h, y^k) + (\lambda_1 y)^{k-m} \cdot (\alpha y)^m + \cdots + (\lambda_4 y)^{k-q} \cdot (\delta y)^q \equiv 0 \qquad (h + k = l),$$

valid when x is a common point of the four corresponding surfaces, to obtain pure resultants $D = R$. However their exposition was confined to those cases in which the unknown coefficients $(\lambda_i y)$ in (I) were unique, i. e., when $k < m + n$. It was also confined to those cases in which additional equations

obtained from the dialytic array did not overlap, i. e., when $h < m + n$. Thus $h + k = l = m + n + p + q - 4 < 2m + 2n - 1$ or $p + q - n \leq m + 2$. In view of the inequalities (24) the cases thus covered are merely

$$
\begin{aligned}
m, n, p, q &= m, m, m, m + r \ (r = 0, 1, 2); \\
&= m, m, m + 1, m + 1; \\
&= m, m + r, m + r, m + r \ (r = 1, 2); \\
&= m, m + 1, m + 1, m + 2.
\end{aligned} \tag{25}
$$

9. Quaternary elimination with $D' = R \cdot A^q$. Following the new ternary procedure developed above we now permit the overlapping of α with β, or γ, or δ, but not that of β with γ. This leads to the restrictions $h < n + p$ and $k < n + p$. These, in conjunction with (24), yield the following cases:

$$m \leq n \leq p \leq q \leq n + p - m + 2. \tag{26}$$

We give only one determinant to cover these cases rather than the variety used in the ternary case. We use the identity

$$(\mathrm{I}) \quad J(x^{m+q-3}, y^{n+p-1}) + (\lambda_1 y)^{n+p-m-1} \cdot (\alpha y)^m + \cdots + (\lambda_4 y)^{n+p-q-1} (\delta y)^q \equiv 0,$$

and form the determinant D' for $z = E_1 = 1, 0, 0, 0$. We shall reduce the number of coefficients in $(\lambda_2 y)^{p-1}$, $(\lambda_3 y)^{n-1}$, $(\lambda_4 y)^{n+p-q-1}$ by requiring the vanishing of all the coefficients which appear in the m-th polar of z. Then the identity (I) has unique coefficients λ. We also add the dialytic identities,

$$(\mathrm{III}) \quad (\sigma_1 x)^{q-3} \cdot (\alpha x)^m \equiv 0 \ (\text{in } \sigma_1), \cdots, (\sigma_4 x)^{m-3} \cdot (\delta x)^q \equiv 0 \ (\text{in } \sigma_4),$$

with again the restriction that the coefficients of $(\sigma_2 y)^{m+q-n-3}$, $(\sigma_3 y)^{m+q-p-3}$ which appear in the m-th polar of z are themselves zero. Then the dialytic identities (III) yield independent equations. After these preliminaries the elimination of the combinations x of degree $m + q - 3$ and of the remaining coefficients λ yields a determinant D' of the form:

(27)

	$a(x$'s$)$	$a_1(\lambda_1$'s$)$	$a_2(\lambda_2$'s$)$	$a_3(\lambda_3$'s$)$	$a_4(\lambda_4$'s$)$
(I) : b	$\alpha\beta\gamma\delta$	α	β	γ	δ
(IIIa) : b_1	α	0	0	0	0
(IIIb) : b_2	β	0	0	0	0
(IIIc) : b_3	γ	0	0	0	0
(IIId) : b_4	δ	0	0	0	0 .

The numbers, a, of columns and, b, of rows are as follows:

$$
\begin{aligned}
&a = [m+q]_3, && b = [n+p+2]_3,\\
&a_1 = [n+p-m+2]_3, && b_1 = [q]_3,\\
(28)\quad &a_2 = [p+2]_3 - [p-m+2]_3, && b_2 = [m+q-n]_3 - [q-n]_3,\\
&a_3 = [n+2]_3 - [n-m+2]_3, && b_3 = [m+q-p]_3 - [q-p]_3,\\
&a_4 = [n+p-q+2]_3 && b_4 = [m]_3.\\
&\quad - [n+p-q-m+2]_3,
\end{aligned}
$$

It can now be verified that

$$a + a_1 + \cdots + a_4 = b + b_1 + \cdots + b_4.$$

The degree in the coefficients of $J(x^h, y^k)$ is

$$j = b - a_1 - \cdots - a_4 = a - b_1 - \cdots - b_4.$$

The degrees of the expansion of D' in the coefficients α, β, γ, δ are respectively $npq + d$, mpq, mnq, mnp, where

$$
\begin{aligned}
(29)\quad d = {} & [p-m+2]_3 + [n-m+2]_3 + [n+p-q-m+2]_3\\
& + [q-n]_3 + [q-p]_3.
\end{aligned}
$$

Then the usual verification that $D' \not\equiv 0$ yields the theorem:

(30) *If* $(\alpha y)^m = A y_1^m + \cdots$ *with* $A \not\equiv 0$, *the determinants* D' *indicated in* (27) *is* $R \cdot A^d$ *with* d *as in* (29).

Naturally if z had been chosen generically and the determinant D' had been replaced by a determinant D formed as in the ternary case, the value of D would be $R \cdot [(\alpha z)^m]^d$.

10. Quaternary elimination in the general case. We are here concerned with orders

$$(31)\qquad m \leq n \leq p \leq q \geq n + p - m + 3,$$

these being the cases not covered in **8** and **9**. In order to keep the dialytic equations as simple as possible we shall take $h = m + n - 1$. The identity (I) then reads:

$$\text{(I)}:\quad J(x^{m+n-1}, y^{p+q-3}) + (\lambda_1 y)^{p+q-m-3} \cdot (\alpha y)^m + \cdots + (\lambda_4 y)^{p-3} \cdot (\delta y)^q \equiv 0.$$

Since λ_2, λ_3, λ_4 can be affected by multiples of $(\alpha y)^m$, we restore their uniqueness in this respect by the requirement that

$$
\begin{aligned}
\text{(II)}:\quad & (\lambda_2 y)^{p+q-n-m-3}(\lambda_2 z)^m \equiv 0,\ (\lambda_3 y)^{q-m-3}(\lambda_3 z)^m \equiv 0,\\
& (\lambda_4 y)^{p-m-3}(\lambda_4 z)^m \equiv 0.
\end{aligned}
$$

Since λ_3, λ_4 can also be affected by multiples of $(\beta y)^n$ we restore their uniqueness in this respect by the requirement that

$$\text{(III)}: \quad \begin{aligned} (\lambda_3 y)^{q-n-3}(\lambda_3 z')^n + (\lambda_{23} y)^{q-m-n-3}\cdot(\alpha y)^m &\equiv 0, \\ (\lambda_4 y)^{p-n-3}(\lambda_4 z')^n + (\lambda_{24} y)^{p-m-n-3}\cdot(\alpha y)^m &\equiv 0. \end{aligned}$$

Finally we add the dialytic equations,

$$\text{(IV)}: \quad (\eta x)^{n-1}\cdot(\alpha x)^m \equiv 0, \quad (\eta x)^{m-1}\cdot(\beta x)^n \equiv 0,$$

on the assumption that $p \geqq m+n$. This assumption is by no means necessary but it is more general than $p < m+n$ and it leads at least to the exclusion of $(\gamma x)^p$ from the dialytic equations and thus yields a more specific case.

The general effect of restricting h to $m+n-1$ is to throw the inherent difficulty of the problem entirely upon the form of the conditions which make the coefficients $\lambda_1, \cdots, \lambda_4$ in (I) unique. We are using here the following property of a module of j forms in j variables with $R \neq 0$: If z, z', z'', $\cdots$ are arbitrarily chosen value systems of y, the form,

$$(\lambda_1 y)^{k-l_1}\cdot(\alpha_1 y)^{l_1} + (\lambda_2 y)^{k-l_2}\cdot(\alpha_2 y)^{l_2} + \cdots + (\lambda_j y)^{k-l_j}\cdot(\alpha_j y)^{l_j},$$

belonging to the module has unique coefficients $\lambda_1, \lambda_2, \cdots, \lambda_j$ if these coefficients are subject to the conditions that the l_1-th polars of z as to $\alpha_2, \cdots, \alpha_j$ vanish identically, the l_2-th polars of z' as to $\alpha_3, \cdots, \alpha_j$ are in the module determined by $(\alpha y)^{l_1}$, the l_3-th polars of z'' as to $\alpha_4, \cdots, \alpha_j$ are in the module determined by $(\alpha_1 y)^{l_1}$, $(\alpha_2 y)^{l_2}$, etc. We mention this general property here partly because we are using a very special case of it and partly because the following memoir contains a radically different method for obtaining unique coefficients λ.

With the equations derived from the identities (I), (II), (III), (IV) we eliminate the combinations x^h, and the coefficients $\lambda_1, \cdots, \lambda_4, \lambda_{23}, \lambda_{24}$ to obtain the determinant D,

(32)

	$[m+n+2]_3$ x	$[q+p-m]_3$ λ_1	$[q+p-n]_3$ λ_2	$[q]_3$ λ_3	$[p]_3$ λ_4	$[q-m-n]_3$ λ_{23}	$[p-$ $\lambda_$
$[q+p]_3$	$\alpha\beta\gamma\delta$	α	β	γ	δ	0	
$[q+p-m-n]_3$	0	0	z^m	0	0	0	
$[q-m]_3$	0	0	0	z^m	0	0	
$[p-m]_3$	0	0	0	0	z^m	0	
$[q-n]_3$	0	0	0	$z^{n'}$	0	α	
$[p-n]_3$	0	0	0	0	z'^n	0	
$[n+2]_3$	α	0	0	0	0	0	
$[m+2]_3$	β	0	0	0	0	0	

The degrees in the coefficients of $(\gamma y)^p$, $(\delta y)^q$ of this determinant are mnq and mnp respectively. The degrees in the coefficients of $(\alpha y)^m$, $(\beta y)^n$ are respectively $npq + d$ and $mpq + e$ where

$$(33)\quad \begin{aligned} d &= [q + p - m - n]_3 + [q - m]_3 + [p - m]_3 \\ e &= [q - n]_3 - [q - n - m]_3 + [p - n]_3 - [p - n - m]_3. \end{aligned}$$

Thus we might expect D to be $R \cdot \{[(\alpha z)^m]^d \cdot [(\beta z')^n]^e + \cdots\}$. We shall actually determine the nature of this factor, extraneous to R, in D and thus prove incidentally that $D \neq 0$ and that its significant factor is R.

Since we may, and will henceforth, take z, z' to be

$$z = 1, 0, 0, 0; \quad z' = 0, 1, 0, 0$$

the determination of the extraneous factor E is essentially a binary problem which can be accomplished in the particular case,

$$\begin{aligned} (\alpha y)^m &= A_0 y_1^m + [m]_1 A_1 y_1^{m-1} y_2 + \cdots + A_m y_2^m \quad (A_0 \neq 0), \\ (\beta y)^n &= B_0 y_1^n + [n]_1 B_1 y_1^{n-1} y_2 + \cdots + B_n y_2^n, \\ (\gamma y)^p &= y_3^p, \\ (\delta y)^q &= y_4^q. \end{aligned}$$

If, for this particular case, the value of E is determined in terms of A_i, B_j its value for D in (32) will be obtained by setting

$$(34)\qquad A_i = (\alpha z)^{m-i}(\alpha z')^i, \qquad B_j = (\beta z)^{n-j}(\beta z')^j.$$

In general we arrange terms $y_1^{i_1} y_2^{i_2} y_3^{i_3} y_4^{i_4}$ (or corresponding coefficients or equations) first with reference to descending i_1, and, for equal i_1's, with reference to descending i_2, etc. We refer to this briefly as a *canonical arrangement*.

Let us first trim off outlying matrices from D. The simplest ones to dispose of are those in z^m. With $z = 1, 0, 0, 0$, the identities (II) merely assert that all terms in λ_2, λ_3, λ_4 with $i_1 \geq m$ have zero coefficients so that our first reduction can be described as

$$R_1\colon\ \lambda_2, \lambda_3, \lambda_4 \text{ have only terms for which } i_1 < m.$$

We consider now the matrices in z'^n and α under λ_4 and λ_{24} respectively derived from the $[p - n]_3$ equations implied by the second identity in (III). These equations divide into two classes: (a) those for which $i_1 \geq m$, and (b) those for which $i_1 < m$. In the equations (a) the coefficients λ_4 are zero due to R_1 and a particular equation must contain a term of $(\lambda_{24} y)^{p-n-m-3} \cdot A_0 y_1^m$.

If then the equations (a) are arranged canonically, they will have under λ_{24}, also in canonical arrangement, a determinant with a diagonal made up of elements A_0 and with zeros to the right of the diagonal. This determinant, $A_0^{[p-n-m]_3}$, is a factor of D. With the matrix α on the right thus eliminated, the remaining equations (b) merely require the vanishing of those coefficients of λ_4 for which $i_2 \geq n$. The same considerations apply to the matrices in z'^n and α under λ_3 and λ_{23} respectively so that, for both sets of matrices, we acquire an outstanding factor of D,

$$E_1 = A_0^{[p-n-m]_3+[q-n-m]_3},$$

and have a reduction:

$$R_2: \begin{array}{l} \lambda_2 \text{ has only terms for which } i_1 < m; \\ \lambda_3, \lambda_4 \text{ have only terms for which } i_1 < m \text{ and } i_2 < n. \end{array}$$

The determinant which results from these reductions has the form,

$$D' = \begin{array}{r|ccccc} & [m+n+2]_3 & [q+p-m]_3 & a_2 & a_3 & a_4 \\ & x & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \hline [q+p]_3 & \alpha\beta\gamma\delta & \alpha & \beta & \gamma & \delta \\ [n+2]_3 & \alpha & 0 & 0 & 0 & 0 \\ [m+2]_3 & \beta & 0 & 0 & 0 & 0, \end{array}$$

where

$$a_2 = [q+p-n]_3 - [q+p-n-m]_3,$$
$$a_3 = [q]_3 - [q-m]_3 - [q-n]_3 + [q-m-n]_3 = mn[q-1-(m+n)$$
$$a_4 = [p]_3 - [p-m]_3 - [p-n]_3 + [p-m-n]_3 = mn[p-1-(m+n)$$

The $[q+p]_3$ equations derived from the identity (I) may be characterized by i_1, i_2, i_3, i_4, (the coefficient of $y_1^{i_1}y_2^{i_2}y_3^{i_3}y_4^{i_4}$ in (I)) where $i_1+i_2+i_3+i_4 = q+p-3$. We isolate from these two sets of equations characterized as follows:

$$\text{Eq. (A)}: i_3 \geq p,\ i_1 \leq m-1,\ i_2 \leq n-1;$$
$$\text{Eq. (B)}: i_4 \geq q,\ i_1 \leq m-1,\ i_2 \leq n-1.$$

The two sets do not collide because $i_3 + i_4 \leq p+q-3$. In the identity (I) for our special case, the terms in $\lambda_3^{q-3} \cdot y_3^p + \lambda_4^{p-3} \cdot y_4^q$ occur in one and only one of these equations, and each of the equations contains one and only one coefficient of either λ_3 or λ_4. No other equations contain coefficients of λ_3 or λ_4. Thus the matrices under λ_3, λ_4 contribute numerical factors of D' which

may be dropped if λ_3, λ_4 and Eq. (A), (B) are dropped. Then D' reduces to D'' where

$$D'' = \begin{array}{r|ccc} & [m+n+2]_3 & [q+p-m]_3 & a_2 \\ & x & \lambda_1 & \lambda_2 \\ \hline [q+p]_3 - a_3 - a_4 & \alpha\beta\gamma\delta & \alpha & \beta \\ [n+2]_3 & \alpha & 0 & 0 \\ [m+2]_3 & \beta & 0 & 0 \end{array} = D/E_1.$$

We proceed now to account for the matrices in the first columns of D'' by isolating a set of equations (C) from the identity characterized as follows:

$$\text{Eq. (C)}: \; i_1 + i_2 \leq m+n-2, \; i_3 \leq p-1, \; i_4 \leq q-1.$$

These are the only equations which involve the coefficients of $J(x^{m+n-1}, y^{p+q-3})$. For $J(y^{m+n+p+q-4})$ has, in our special case, the value,

$$J(y) = \begin{vmatrix} (\alpha y)^{m-1}\alpha_1 & (\alpha y)^{m-1}\alpha_2 \\ (\beta y)^{n-1}\beta_1 & (\beta y)^{n-1}\beta_2 \end{vmatrix} y_3^{p-1} y_4^{q-1}.$$

Since $(\alpha y)^m$, $(\beta y)^n$ are binary forms in y_1, y_2 alone, the highest degree in y_1, y_2 which can be attained after polarization as to x is $m+n-2$, and this and all lower degrees in y_1, y_2 occur in $J(x, y)$. We note also that the combinations of x_1, x_2 to the degree $m+n-1$ will not occur in $J(x, y)$. These $m+n$ combinations arise from the matrices α, β below J by using the identities (IV) for $\xi_3 = \xi_4 = 0$. This particular set of $m+n$ equations in these $m+n$ combinations has for determinant Sylvester's dialytic determinant, the resultant

$$r = r(A^m, B^n)$$

of the binary forms, $A_0 y_1^m + \cdots, B_0 y_1^n + \cdots$. Thus r^1 is a factor of D''.

In order to handle the remaining dialytic equations and the equations C we recall the $m+n-1$ various forms of the binary eliminant of $(\alpha y)^m$, $(\beta y)^n$ which involve the coefficients of their jacobian [cf. [5]]. These can be tabulated in the following typical sequence:

$$1^0: \; J(x^{m+n-2}) = 0, \qquad \begin{array}{r|c} & x^{m+n-2} \\ \hline 1 & \alpha\beta \\ n-1 & \alpha \\ m-1 & \beta \end{array} = r;$$

$$(\xi x)^{n-2} \cdot (\alpha x)^m \equiv 0, \; (\xi x)^{m-2} \cdot (\beta x)^n \equiv 0,$$

. . . . . .

$$(m-1)^0\colon J(x^n, y^{m-2}) \equiv 0, \qquad \begin{array}{c|c} & x^n \\ \hline m-1 & \alpha\beta \\ n-m+1 & \alpha \\ 1 & \beta \end{array} = r;$$
$$(\xi x)^{n-m}\cdot(\alpha x)^m \equiv 0,\ (\beta x)^n = 0,$$

$$m^0\colon J(x^{n-1}y^{m-1}) \equiv 0, \qquad \begin{array}{c|c} & x^{n-1} \\ \hline m & \alpha\beta \\ n-m & \alpha \end{array} = r;$$
$$(\xi x)^{n-m-1}\cdot(\alpha x)^m \equiv 0,$$

$$(m+1)^0\colon J(x^{n-2}, y^m) + \lambda(\alpha y)^m \equiv 0, \qquad \begin{array}{c|cc} & x^{n-2} & \lambda^0 \\ \hline m+1 & \alpha\beta & \alpha \\ n-m-1 & \alpha & 0 \end{array} = r;$$
$$(\xi x)^{n-m-2}\cdot(\alpha x)^m \equiv 0,$$

. . . . . .

$$(n-1)^0\colon J(x^m, y^{n-2}) + (\lambda y)^{n-m-2}\cdot(\alpha y^m \equiv 0, \qquad \begin{array}{c|cc} & x^m & \lambda^{n-m-2} \\ \hline n-1 & \alpha\beta & \alpha \\ 1 & \alpha & 0 \end{array} = r;$$
$$(\alpha x)^m = 0,$$

$$n^0\colon J(x^{m-1}, y^{n-1}) + (\lambda y)^{n-m-1}\cdot(\alpha y)^m \equiv 0, \qquad \begin{array}{c|cc} & x^{m-1} & \lambda^{n-m-1} \\ \hline n & \alpha\beta & \alpha \end{array} = r;$$

$$(n+1)^0\colon J(x^{m-2}, y^n) + (\lambda y)^{n-m}\cdot(\alpha y)^m + \mu(\beta y)^n \equiv 0,$$

$$\begin{array}{c|ccc} & x^{m-2} & \lambda^{n-m} & \mu^0 \\ \hline n+1 & \alpha\beta & \alpha & \beta \end{array} = r;$$

. . . . . .

$$(n+m-1)^0\colon J(y^{m+n-2}) + (\lambda y)^{n-2}\cdot(\alpha y)^m + (\mu y)^{m-2}\cdot(\beta y)^n \equiv 0,$$

$$\begin{array}{c|ccc} & 1 & \lambda^{n-2} & \mu^{m-2} \\ \hline m+n-1 & \alpha\beta & \alpha & \beta \end{array} = r.$$

The binary identity 1^0 contains all combinations of x_1, x_2 to the degree $m+n-2$. To bring these up to the degree $m+n-1$ of J in x we can multiply them by x_3 or by x_4. If x_3 is the factor chosen for all of the equations 1^0, then those arising from J must have a unique further factor $y_3^{p-2}y_4^{q-1}$ in order to produce equations (C); if x_4 is chosen, the further

factor is $y_3^{p-1}y_4^{q-2}$. Corresponding to these two factors we have two sets of equations 1^0 which have determinants r and the case 1^0 contributes a factor r^2 of D''. In the next identity 2^0 we must use factors $x_3^2y_3^{p-3}y_4^{q-1}$, $x_3x_4y_3^{p-2}y_4^{q-2}$, $x_4^2y_3^{p-1}y_4^{q-3}$, and thus obtain a factor r^3. Proceeding in this way up to $(m+n-1)^0$ we obtain a factor

$$E_2 = r^{1+2+3+\ldots+(m+n)} = r^{[n+m+1]_2}$$

of D.

In the above process we have used all of the equations (C) and all of the dialytic equations obtained from the identities (IV) and have thus exhausted the matrices in the first column of D. Of the variables we have eliminated all of the x^{m+n-1}. In addition, however, from $(m+1)^0$ on we have eliminated coefficients λ_1 (as binary coefficients λ) and coefficients λ_2 (as binary coefficients μ). The coefficients of $(\lambda_1 y)^{q+p-m-3}$ thus eliminated are characterized by

$$i_1 + i_2 \leq n-2,\ i_3 \leq p-1,\ i_4 \leq q-1, \tag{35}$$

and they number

$$\begin{aligned} b_1 &= (m+2)\cdot 1 + (m+3)\cdot 2 + \cdots + (m+n)\cdot(n-1) \\ &= m[n]_2 + 2[n+1]_3; \end{aligned}$$

those of $(\lambda_2 y)^{q+p-n-3}$ eliminated are characterized by

$$i_1 + i_2 \leq m-2,\ i_3 \leq p-1,\ i_4 \leq q-1, \tag{36}$$

and they number

$$\begin{aligned} b_2 &= (n+2)\cdot 1 + (n+3)\cdot 2 + \cdots + (n+m)(m-1) \\ &= n[m]_2 + 2[m+1]_3. \end{aligned}$$

The remaining equations and variables have a determinant of the form,

$$D''' = c \begin{vmatrix} c_1(\lambda_1\text{'s}) & c_2(\lambda_2\text{'s}) \\ \alpha & \beta \end{vmatrix} = D/E_1E_2,$$

where

$$\begin{aligned} c_1 &= [q+p-m]_3 - b_1, \\ c_2 &= a_2 - b_2, \\ c &= [q+p]_3 - a_3 - a_4 - b_1 - b_2 - [m+n+2]_3 \\ &\quad + [n+2]_3 + [m+2]_3. \end{aligned}$$

The equations are those which arise from the identity (I) except (A), (B), (C). However, for these the identity (I) now takes the simple form

$$(\mathrm{I}'): \quad (\lambda_1 y)^{q+p-m-3} \cdot (\alpha y)^m + (\lambda_2 y)^{q+p-n-3} \cdot (\beta y)^n \equiv 0.$$

In this all of the coefficients of λ_1 and λ_2 are present except those excluded by (C) as described in (35), (36) and also except that λ_2 contains terms in ${y_1}^0, {y_1}^1, \cdots, {y_1}^{m-1}$ only. Consider then a new set of equations (D) derived from (I′) and characterized by

$$\text{Eq. (D)}: \quad i_1 \geq m + n.$$

These contain only the coefficients λ_1 in (I′). If then both the equations and the coefficients λ_1 which they contain are arranged in the canonical order their determinant will have elements A_0 on the diagonal and zeros to the right of the diagonal. The number of these equations is

$$[2]_2 + [3]_2 + \cdots + [q + p - 1 - m - n]_2 = [q + p - m - n]_3.$$

Thus D''' contains a factor,

$$E_3 = {A_0}^{[q+p-m-n]_3}.$$

The remaining equations from I′ are those, exclusive of (A), (B), (C) which would be obtained from an identity of the form,

(I″):

$$[(\lambda_{1,0} y)^{q+p-m-n-2} {y_1}^{n-1} + \cdots + (\lambda_{1,n-1} y)^{q+p-m-3}][A_0 {y_1}^m + \cdots + A_m {y_2}^m]$$
$$+ [(\lambda_{2,0}) y^{q+p-m-n-2} {y_1}^{m-1} + \cdots + (\lambda_{2,m-1} y)^{q+p-n-3}][B_0 {y_1}^n + \cdots + B_n {y_2}^n] \equiv 0,$$

where the $\lambda_{i,j}$ are forms in y_2, y_3, y_4 with coefficients excluded by (35), (36) omitted.

In the permissible equations derived from (I″) let i_3, i_4 be fixed. Then, in the forms $\lambda_{i,j}$, the exponent of y_2 is also fixed. Thus we have, for $i_1 + i_2 = q + p - 3 - i_3 - i_4$ and $i_1 = 0, \cdots, m + n - 1$, precisely $m + n$ equations to fix the values of the $m + n$ coefficients $\lambda_{i,j}(i_1, i_2, i_3, i_4)$. These $m + n$ coefficients will appear in no other equations and their determinant is the resultant, r. This factor will appear for all sets of $m + n$ equations from $i_1 + i_2 = m + n - 1$ up to $i_1 + i_2 = q + p - 3$ with corresponding $i_3 + i_4$ descending from $q + p - m - n - 2$ down to 0. Thus the number of such sets of $m + n$ equations (i. e., the number of sets i_3, i_4) is

$$1 + 2 + \cdots + (q + p - m - n - 1) = [q + p - m - n]_2.$$

We have therefore another factor of D''', namely

$$E_4 = r^{[q+p-m-n]_2}.$$

Consider now the equations from (I″) for fixed i_3, i_4 when $i_1 + i_2 = m + n - 2$. We lose two variables, one from $(\lambda_{1,0} y)$ and one from $(\lambda_{2,0} y)$.

But we also lose two equations, one because the range of i_1 from 0 to $m+n-2$ is reduced, and one from the combination $i_1, i_2 = m-1, n-1$. Indeed this latter equation is barred by (A) when $i_3 \geq p$, by (B) when $i_4 \geq q$, and by (C) for intermediate values. Thus we have, for fixed i_3, i_4 precisely $m+n-2$ equations in $m+n-2$ coefficients whose determinant is

$$r_1 = r_1(A^{n-1}, B^{m-1}),$$

a *first sub-resultant of* r, obtained by dropping two rows and two columns of r in the dialytic form of r. The number of pairs i_3, i_4 which contribute such a factor r_1 of D''' is the number not excluded by (C), i. e., $p+q-2(m+n)$.

When $i_1 + i_2 = m+n-3$, we lose two equations because of the range for i_1, and two equations for $i_1, i_2 = m-1, n-2$; $m-2, n-1$, and thus have, for fixed i_3, i_4, $m+n-4$ equations in $m+n-4$ coefficients whose determinant is

$$r_2 = r_2(A^{n-2}, B^{m-2}),$$

a *second sub-resultant of* r. Since the value of $i_3 + i_4$ is now increased by one we have $p+q-2(m+n-1)$ admissible pairs i_3, i_4. This process goes on until $i_1 + i_2 = n$, and we get a subresultant $r_{m-1}(A^{n-m+1}, B^1)$ for each of $p+q-2(n+2)$ pairs i_3, i_4. Thus we have a further factor of D''' namely,

$$E_5 = r_1^{p+q-2(m+n)} r_2^{p+q-2(m+n-1)} \cdots r_{m-1}^{p+q-2(n+2)}.$$

When $i_1 + i_2 = n-1$, r_m is merely A_0^{n-m} taken $p+q-2(n+1)$ times. Thereafter the number of equations lost is due only to the decreasing range of i_1. Finally, with $i_1 + i_2 = m$, we reach A_0^1 taken $p+q-2m-4$ times. When $i_1 + i_2 \leq m-1$ the equations are all excluded by (A), (B), or (C). Thus we have a further final factor of D''', namely,

$$E_6 = A_0^f,$$

where

$$\begin{aligned} f &= (n-m)(p+q-2n-2) + (n-m-1)(p+q-2n) \\ &\qquad + \cdots + (1)(p+q-2m-4) \\ &= (p+q-2m-2)[n-m+1]_2 \\ &\qquad - (n-m)(n-m+1)(2n-2m+1)/3. \end{aligned}$$

We have thus evaluated the deterimnant D for the special case here used in the form,

$$D = E_1 E_2 E_3 E_4 E_5 E_6.$$

In this special case, however, R, the resultant of the four forms, is r^{pq}. This factor r appears in E_2 and E_4 so that we have finally

(37) $$D = R \cdot A_0^g r^h E_5,$$

where

$$g = f + [q - n - m]_3 + [p - n - m]_3 + [q + p - n - m]_3,$$
$$h = [q + p - m - n]_2 + [n + m + 1]_2 - pq.$$

We may state the above final result as follows:

(38) *The determinant D in (32) is the resultant R of the four given forms multiplied by an extraneous factor which is a product of powers of $(\alpha z)^m$, and of powers of the resultant and subresultants of two binary forms of orders m, n with coefficients (34).*

As a check on the entire procedure we had already found in (33) the degrees d, e of the extraneous factor in the coefficients of $(\alpha y)^m$, $(\beta y)^n$. Now the degree of the above extraneous factor $A_0^g r^h E_5$ in the coefficients of $(\alpha y)^m$ is

$$\begin{aligned} g + nh + \{&(n - m + 1)(p + q - 2n - 4) \\ &+ (n - m + 2)(p + q - 2n - 6) + \cdots \\ &+ (n - 1)(p + q - 2n - 2m)\} \\ = g + nh &+ (p + q - 2n - 2)\{(m - 1)(n - m) + [m]_2\} \\ &- m(m - 1)(2m - 1)/3 - 2(n - m)[m]_2, \end{aligned}$$

and this is equal to d in (33). Similarly the degree of $r^h E_5$ in the coefficients of $(\beta y)^n$ turns out to be e in (33).

11. Elimination in higher domains. In the Morley-Coble account [4], the restriction (a), $h \leq m + n - 1$, $k \leq m + n - 1$, barred all identities of the sort, $(\beta y)^n \cdot (\alpha y)^m \equiv (\alpha y)^m \cdot (\beta y)^n$, and yielded a pure resultant R in the cases covered. These cases comprised arbitrary orders m, n in the ternary domain, and arbitrary m in the quaternary domain. In the present account $(1, \cdots, 9)$ the restrictions (b), $h \leq n + p - 1$, $k \leq n + p - 1$ admit identities of the above sort only between $(\alpha y)^m$ and any one of the remaining forms. The cases then covered comprised arbitrary m, n, p in both the ternary and quaternary domains. Finally in **10** the restrictions (c), $h \leq p + q - 1$, $k \leq p + q - 1$ admit identites of the above sort between $(\alpha y)^m$ or $(\beta y)^n$ and any one of the remaining forms. The cases then covered exhausted the quarternary domain.

Nevertheless the flexibility of the method rapidly diminishes. The essential reason for this is that, in j variables, we have

$$h + k = l = l_1 + \cdots + l_j - j,$$

and, for given l, there is only one degree of freedom in the choice of h, k whereas there are $j-1$ degrees in the choice of the orders $l_1, \cdots, l_j$. Thus, in the quinary domain with $l_1, \cdots, l_5 = 4, 5, 5, 5, 5$; $l = 19$, the most favorable choice of h, k is 9, 10 and then all of the possible overlapping identities appear. This situation would require the introduction of four parametric points z, z', z'', z''' as explained in **10**. If, as further parameters are necessary, some well defined pattern for the extraneous factor would appear, it might be worth while to pursue the method more generally. Our main reason for the determination of E in **10** was not to complete definitely the ternary and quaternary elimination but rather to show that such a pattern is not to be expected.

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ON THE EXPRESSION OF AN ALGEBRAIC FORM IN TERMS OF A SET OF FORMS WITH NON-ZERO RESULTANT.*

By ARTHUR B. COBLE

1. Introduction. Given n algebraic forms in n variables,

$$f_i = (\alpha_i y)^{l_i} \quad (i = 1, \cdots, n), \quad (l_1 \leq 2l_2 \leq \cdots \leq l_n),$$

we set

$$l = \Sigma_i (l_i - 1), \qquad L_i = (\prod_{j=1}^{j=n} l_j)/l_i.$$

If the resultant R of the n given forms does not vanish, it is well known that an algebraic form of order $m > l$ can be expressed as a member of the module determined by the given forms, i. e., that

$$R \cdot (\delta y)^m \equiv \sum_{i=1}^{i=n} (\beta_i y)^{m-l_i} \cdot (\alpha_i y)^{l_i}. \tag{1}$$

Ordinarily (1) is indicated by the notation $R \cdot (\delta y)^m \equiv 0 \pmod{f_1, \cdots, f_n}$, with no attention to the coefficients, $(\beta_i y)^{m-l_i}$, in the expression (1). It is the purpose of this article to determine explicit values for these coefficients as covariants of the given forms and of $(\delta y)^m$. We also determine an explicit extension of the module defined by the given forms which will yield an expression of type (1) for orders $0 < m \leq l$. In these expressions the resultant R on the left of (1) will be balanced on the right by the coefficients of the unique form, $A = (a\eta)^l$, apolar to the n forms f_i which define the module. This balance is accentuated by the fact that, in the generic case, *neither R nor A* is explicitly known. Thus an independent discussion of the form A and of some of its properties may be in order. Again we find it convenient to denote the binomial coefficients by the symbol

$$[n]_k = {}_nC_k.$$

2. The unique form A apolar to the given forms. Let $C_{l_1}(r)$ be the number of linearly independent conditions imposed on a form $F = (b\eta)^r$ of class r by its apolarity with a given form, $f_1 = (\alpha_1 y)^{l_1}$, of order l_1, i. e., by the identical vanishing of $(b\alpha_1)^{l_1}(b\eta)^{r-l_1}$. Let $C_{l_1, \ldots, l_k}(r)$ be the number of linearly independent conditions imposed on F by its apolarity with k given generic forms $f_1, f_2, \cdots, f_k$. We wish to prove that

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$$(2) \qquad C_{l_1,\ldots,l_k}(r) = \Sigma C_{l_1}(r) - \Sigma C_{l_1+l_2}(r) + \Sigma C_{l_1+l_2+l_3}(r) - \cdots + (-1)^{k-1} C_{l_1+\ldots+l_k}(r).$$

where the summations refer to all of the combinations indicated by the subscripts which can be drawn from $l_1, \cdots, l_k$.

We observe that the value (2) of $C_{l_1,\ldots,l_k}(r)$ can be separated into three parts as follows,

$$(3) \qquad C_{l_1,\ldots,l_k}(r) = C_{l_1,\ldots,l_{k-1}}(r) + C_{l_k}(r) - C_{l_1,\ldots,l_{k-1}}(r-l_k),$$

according as a subscript on the right of (2) does not contain l_k, contains l_k only, or contains l_k together with other l's. This formula (3) is a recursion formula for the determination of $C_{l_1,\ldots,l_k}(r)$ in terms of $C_{l_1,\ldots l_{k-1}}$. It expresses the fact that, of the C_{l_k} conditions imposed on F by its apolarity with f_k, conditions in themselves linearly independent, those which arise from the apolarity of $(b\alpha_k)^{l_k}(b\eta)^{r-l_k}$ with the $k-1$ earlier forms are themselves a consequence of the earlier apolarities. This recursion formula is then the basis for an induction proof of (2) which by definition is valid when $k=1$.

We now make use of an identity among the binomial coefficients, namely:

$$(4) \qquad [l_1 + \cdots + l_n - 1]_{n-1} - \Sigma[l_1 + \cdots + l_{n-1} - 1]_{n-1} + \Sigma[l_1 + \cdots + l_{n-2} - 1]_{n-1} - \cdots (-1)^{n-1}\Sigma[l_1 - 1]_{n-1} + (-1)^n[-1]_{n-1} = 0.$$

All of the terms of this identity cancel due to relations of the form $(1-1)^j = 0 \ (j = 1, \cdots, n)$. The first term of (4) is the number of coefficients of the form $A = (a\eta)^l$ and the last term is -1. The intermediate terms by comparison with (2) are $-C_{l_1,\ldots,l_n}(l)$. Hence

(5) *If $f_i = (\alpha_i y)^{l_i}$ $(i = 1, \cdots, n)$ are n generic forms of the orders indicated in variables $y_1, \cdots, y_n$, there is a single form $A = (a\eta)^l$ of class $l = l_1 + \cdots + l_n - n$ which is apolar to the n given forms.*

That the conditions, which have not been barred above as obviously dependent on earlier conditions, are themselves linearly independent is evident from the very special, but sufficiently generic, case:

$$(6) \qquad f_i = y_i^{l_i} \ (i = 1, \cdots, n).$$

For, then, $A = c\eta_1^{l_1-1}\eta^{l_2-1} \cdots \eta_n^{l_n-1}$, and the uniqueness is evident.

In line with similar definitions for the resultant of the forms f_i we may state that

(7) *For given forms,* $(\alpha_i y)^{l_i}$ $(i = 1, \cdots, n)$, *the unique apolar form* $A = (a\eta)^l$ *is the* G. C. D. *of the determinants of order* $[l - 1]_{n-1}$ *formed from the array of linear equations in the coefficients* a *obtained from the apolarity conditions,* $(\alpha_i a)^{l_i}(a\eta)^{l-l_i} \equiv 0$, *and the additional linear equation,* $(a\eta)^l = 0$.

Indeed, all of the determinants formed from the identities alone must vanish since the coefficients of $(a\eta)^l$ are not all zero. Thus the non-zero determinants must involve the equation $(a\eta)^l = 0$. A particular non-vanishing one will therefore be the apolar form multiplied by some factor which contains the coefficients of the given forms and which depends upon the manner of selecting equations from the identities.

Another definition of the apolar form is that A is some one of the non-zero determinants mentioned divided by the G. C. D. of the coefficients of the terms in η.

We now prove the theorem:

(8) *If* $L_i = l_1 l_2 \cdots l_n / l_i$, *the degree of the unique apolar form* $(a\eta)^l$ *in the coefficients of* f_i *is* $L_i - 1$.

We prove this for the case $i = n$, making use of the identity,

$$(9) \quad \begin{aligned} &[l_1 + \cdots + l_{n-1} + m]_{n-1} - \Sigma[l_1 + \cdots + l_{n-2} + m]_{n-1} \\ &\quad + \Sigma[l_1 + \cdots + l_{n-3} + m]_{n-1} - \cdots + (-1)^{n-1}[m]_{n-1} \\ &= L_n \qquad\qquad (m = l_n - 1). \end{aligned}$$

The first term of this identity is the number of linearly independent forms of class l. The remaining terms on the left are $-C_{l_1, \ldots, l_{n-1}}(l)$. Thus the left member is the number of linearly independent forms of class l apolar to $f_1, \cdots, f_{n-1}$. We suppose that this linear system of dimension $L_n - 1$ of forms of class l is exhibited in terms of the coefficients of $f_1, \cdots, f_{n-1}$ with L_n arbitrary linear homogeneous parameters and apply to the system the condition of apolarity with f_n. Since this yields just enough conditions to determine the L_n homogeneous parameters only $L_n - 1$ of the new conditions can be linearly independent. Each being linear in the coefficients of f_n, the choice of $L_n - 1$ independent conditions yields an apolar form of degree $L_n - 1$ in the coefficients of f_n. Thus the G. C. D. of (7) has a degree in the coefficients of f_i which is *at most* $L_i - 1$.

We proceed to exhibit a more specific method for setting up the apolar form with the actual degree $L_n - 1$ in the coefficients of f_n. This is a consequence of the theorem:

(10) *If, in S_{n-1}, $n-1$ primals, $(\alpha_1 y)^{l_1}=0,\cdots,(\alpha_{n-1}y)^{l_{n-1}}=0$, meet in L_n distinct points, $(p_i\eta)=0$, there is in general no linear identity connecting the k-th powers of these points if $k\geq l_1+\cdots+l_{n-1}-(n-1)=l-l_n+1$.*

If k were greater than $l-l_n+1$ and an identity for these k-th powers existed, then, by polarizing it, we would have an identity among the lower powers. Thus it will be sufficient to prove the theorem for $k=l-l_n+1$. For this value of k any one of these powers would be apolar to $f_1,\cdots,f_{n-1}$ and we have just seen that the number of linearly independent forms of this class is precisely L_n. That the $(l-l_n+1)$-th powers of these L_n points are themselves in general linearly independent can be proved from their independence in a particular case. This case is the following:

$$(11)\qquad (\alpha_1 y)^{l_1}=y_1{}^{l_1}-y_n{}^{l_1},\cdots,(\alpha_{n-1}y)^{l_{n-1}}=y_{n-1}{}^{l_{n-1}}-y_n{}^{l_{n-1}}.$$

For this special case the points $(p_i\eta)$ are

$$\epsilon_1{}^{i_1}\eta_1+\cdots+\epsilon_{n-1}{}^{i_{n-1}}\eta_{n-1}+\eta_n=0,$$

where $\epsilon_1,\cdots,\epsilon_{n-1}$ are primitive roots of unity of indices $l_1,\cdots,l_{n-1}$ respectively. We wish to prove that an identity of the form,

$$\sum_{n-1}\lambda_{i_1 i_2\ldots i_{n-1}}(\epsilon_1{}^{i_1}\eta_1+\cdots+\epsilon_{n-1}{}^{i_{n-1}}\eta_{n-1}+\eta_n)^{l_1+\ldots+l_{n-1}-(n-1)}\equiv 0,$$

where the summations for each i_j runs from $i_j=0$ to $i_j=l_j-1$, implies the vanishing of the L_n coefficients λ. This identity yields the following L_n equations in λ:

$$(12)\qquad \sum_{n-1}\lambda_{i_1\ldots i_{n-1}}\epsilon_1{}^{r_1 i_1}\epsilon_2{}^{r_2 i_2}\cdots\epsilon_{n-1}{}^{r_{n-1}i_{n-1}}=0$$

$$[r_1+\cdots+r_{n-1}\leq l_1+\cdots+l_{n-1}-(n-1);\ 0\leq r_j\leq l_j].$$

Consider a fixed value k_{n-1} of i_{n-1} and a fixed value s_{n-1} of r_{n-1}. Then (12) can be written in the form

$$(13)\qquad \sum_{k_{n-1}=0}^{\Sigma k_{m-1}=l_{n-1}}\ \sum_{n-2}\lambda_{i_1\ldots i_{n-2}k_{n-1}}\epsilon_1{}^{r_1 i_1}\epsilon_2{}^{r_2 i_2}\cdots\epsilon_{n-2}{}^{r_{n-2}i_{n-2}}\epsilon_{n-2}{}^{r_{n-2}i_{n-2}}\epsilon_{n-1}{}^{s_{n-1}k_{n-1}}=0.$$

For fixed values of $i_1,\cdots,i_{n-2}$ and $r_1,\cdots,r_{n-2}$, as s_{n-1} changes from 0 to $l_{n-1}-1$ we have in (13) l_n equations in the l_n expressions

$$\sum_{n-2}\lambda_{i_1 i_2\ldots i_{n-2}k_{n-1}}\epsilon_1{}^{r_1 i_1}\cdots\epsilon_{n-2}{}^{r_{n-2}i_{n-2}}$$

for which k_{n-1} varies from 0 to $l_{n-1}-1$. The determinant of this system is $|\epsilon_{n-1}{}^{ij}|\neq 0$. Hence each of these expressions vanishes and the system (12)

of L_n equations in L_n variables may be replaced by the l_{n-1} systems of L_n/l_{n-1} equations in L_n/l_{n-1} variables expressed by

$$\sum_{n-2} \lambda_{i_1 \ldots i_{n-2} k_{n-1}} \epsilon_1^{r_1 i_1} \cdots \epsilon_{n-2}^{r_{n-2} i_{n-2}} = 0. \tag{14}$$

In one of these systems defined by fixing k_{n-1} the summation is for $i_1, \cdots, i_{n-2}$ only, and the L_n/l_{n-1} equations are those for which $r_1 + \cdots + r_{n-2} = l_1 + \cdots + l_{n-2} - (n-2)$ since the value $r_{n-1} = l_{n-1} - 1$ has been used. If now we fix i_{n-2} at k_{n-2}, the same argument shows that

$$\sum_{n-3} \lambda_{i_1 \ldots i_{n-3} k_{n-2} k_{n-1}} \epsilon_1^{r_1 i_1} \cdots \epsilon_{n-3}^{r_{n-3} i_{n-3}} = 0,$$

and finally we find that $\lambda_{k_1 \ldots k_{n-1}} = 0$ for any value of $k_1, \cdots, k_{n-1}$ within the limits of $r_1, \cdots, r_{n-1}$.

We supplement the theorem (10) by the following:

(15) *There is in general precisely one linear identity connecting the $(l - l_n)$-th powers of the L_n points of Theorem* (10).

These powers, $(p_i\eta)^{l-l_n}$, are forms to which $f_1, \cdots, f_{n-1}$ are apolar whence only $L_n - 1$ of them can be linearly independent. For, in the identity (9) with $m = -1$, the first term on the left is the number of linearly independent forms of class $l - l_n$, the last term on the left is $+1$, and the intermediate terms are $-C_{l_1, \ldots, n-1}(l - l_n)$. That in general there is only one such identity is clear from the above special case (11). For, an assumed identity would yield equations like (12), with however the limitation on the r's that

$$r_1 + \cdots + r_{n-1} \leq l_1 + \cdots + l_{n-1} - n,$$

which excludes the extreme case

$$r_1 = l_1 - 1, \; r_2 = l_2 - 1, \cdots, r_{n-1} = l_{n-1} - 1.$$

Thus one, and only one, of the equations like (12) would be missing. Since the rank of the system (12) is L_n the rank of the present system is $L_n - 1$, whence the solution is unique.

Since, according to (10), there is no linear identity connecting the powers, $(p_1\eta)^l$, of the L_n points, these powers may be chosen as the L_n linearly independent forms of class l apolar to $f_1, \cdots, f_{n-1}$. The unique apolar form must then be expressible as

$$A = (a\eta)^l = \mu_1 (p_1\eta)^l + \cdots + \mu_{L_n} (p_{L_n}\eta)^l.$$

Since A is also apolar to $f_n = (\alpha_n y)^{l_n}$,

$$\mu_1 (\alpha_n p_1)^{l_n} (p_1\eta)^{l-l_n} + \cdots + \mu_{L_n} (\alpha_n p_{L_n})^{l_n} (p_{L_n}\eta)^{l-l_n} \equiv 0.$$

However, according to (15), there is just one identity connecting these powers and its coefficients v_i can of course depend only on $f_1, \cdots, f_{n-1}$. Hence $\mu_i(\alpha_n p_i)^{l_n} = v_i$. Thus

$$A = (a\eta)^l = \sum_{i=1}^{i=L_n} [v_i (p_i\eta)^l / (\alpha_n p_i)^{l_n}] \cdot \prod_{i=1}^{i=L_n} (\alpha_n p_i)^{l_n}. \tag{16}$$

(17) *The formula* (16) *furnishes the unique apolar form of n generic forms with a degree $L_n - 1$ in the coefficients of f_n, and in terms of constants v_i, p_{ij} which are determined by $f_1, \cdots, f_{n-1}$ alone.*

Another consequence of the formula (16) is that

(18) *If the n forms $f_1, \cdots, f_n$ are generic except for a common zero at $(p_1\eta) = 0$, then $A = (a\eta)^l = c \cdot (p_1\eta)^l$.*

If this common point exists, it is also on the jacobian $J = (\gamma y)^l$ of the n forms. Thus $(\gamma p_1)^l = 0$, or $(a\eta)^l$ and $(\gamma y)^l$ are apolar. Hence the apolarity invariant $(a\gamma)^l$ vanishes if the n forms have a common zero. This invariant does not vanish identically. Indeed, in the special case (6),

$$(\gamma y)^l = y_1^{l_1 - 1} \cdots y_n^{l_n - 1}, \quad (a\eta)^l = \eta_1^{l_1 - 1} \cdots \eta^{l_n - 1}$$

and these are not apolar. Thus this non-vanishing apolarity invariant of degree L_i in the coefficients of f_i vanishes when the irreducible resultant R of the same degrees vanishes, whence

(19) *The apolarity invariant of the unique apolar form, and of the jacobian, of n forms is the resultant R of the n forms.*

In order to make the numerical constants in A, J, R precise we will suppose that, *in forming the apolarity invariant* $[J, A]$, the form A of class l in $\eta_1, \cdots, \eta_n$ is converted into a differential operator in $\partial/\partial y_1, \cdots, \partial/\partial y_n$ which is applied to J. Then

If the numerical factors of proportionality in A, J and R are so chosen that, when the n given forms are

$$f_1 = a_1 y_1^{l_1} + \cdots, \cdots, f_n = a_n y_n^{l_n} + \cdots$$

$$J = \prod_{i=1}^{i=n} (a_i y_i^{l_i - 1}) + \cdots, \quad A = \prod_{i=n}^{i=1} (a_i^{L_i - 1} \eta_i^{l_i - 1}) + \cdots,$$

$$R = \prod_{i=1}^{i=n} [(l_i - 1)\,!\, a_i^{L_i}] + \cdots,$$

then $[J, A]$ as defined above is R.

Indeed, when the given forms contain only their leading terms, then A, J, and R contain only their leading terms for which R satisfies the

definition ((19) *et seq.*). Since these terms occur also in the generic case, they serve to define A, J, and R uniquely.

An example of some of the above properties of A is furnished by the case of two binary forms. Let

$$f_1 = (\alpha y)^m = (\alpha_1 y) \cdot (\alpha_2 y) \cdot \cdots (\alpha_m y),$$

$$f_2 = (\beta y)^n = (\beta_1 y) \cdot (\beta_2 y) \cdot \cdots (\beta_n y).$$

Then, as is well known, there is, when the $m + n$ linear factors are distinct, an identity connecting their $(m + n - 2)$-th powers of the form,

$$\sum_{i=1}^{i=m} \lambda_i (\alpha_i y)^{m+n-2} + \sum_{j=1}^{j=n} \mu_j (\beta_j y)^{m+n-2} \equiv 0,$$

where the coefficients λ_i, μ_j are, to within sign, the products of the bilinear invariants of every pair of the linear factors other than that one to which the coefficient is attached. Therefore the unique apolar form is

$$A = \sum_{i=1}^{i=m} \lambda_i (\alpha_i y)^{m+n-2} \equiv -\sum_{j=1}^{j=n} \mu_j (\beta_j y)^{m+n-2}.$$

The first form of A will have, in the coefficients λ_i, a common factor $\Pi(\beta_j \beta'_j)$ which may be discarded. A particular coefficient λ_i will have the factor $(\beta\alpha_1)^n \cdots (\beta\alpha_{i-1})^n \cdot (\beta\alpha_{i+1})^n \cdots (\beta\alpha_m)^n$, and thus, according to (17), will have the degree $m - 1$ in the coefficients of A. However, this particular form of A can not be rationalized in terms of the coefficients of f_1. Thus it is to be contrasted with the expressions for A obtained by bordering resultants derived from $J(x^h, y^k)$, these being rational, integral, and of the proper degrees, in the coefficients of both f_1 and f_2.

3. The expression (1) when $m > l$. We turn now to the main purpose of this paper and recall a very general identity proved in the memoir cited of Morley-Coble [cf. p. 481 (30)] which reads as follows:

$$\begin{aligned}(21)\quad & J_k(\alpha_1{}^{l_1}, \cdots, \alpha_n{}^{l_n}) \cdot (\delta x)^m \\ & \qquad + \sum_{i=1}^{i=n} J_k(\alpha_1{}^{l_1}, \cdots, \alpha_{i-1}{}^{l_{i-1}}, \alpha_{i+1}{}^{l_{i+1}}, \cdots, \alpha_n{}^{l_n}, \delta^m) \cdot (\alpha_i x)^{l_i} \\ & \equiv J_{k-m}(\alpha_1{}^{l_1}, \cdots, \alpha_n{}^{l_n}) \cdot (\delta y)^m \\ & \qquad + \sum_{i=1}^{i=n} J_{k-l_i}(\alpha_1{}^{l_1}, \cdots, \alpha_{i-1}{}^{l_{i-1}}, \alpha_{i+1}{}^{l_{i+1}}, \cdots, \alpha_n{}^{l_n}, \delta^m) \cdot (\alpha_i y)^{l_i}.\end{aligned}$$

In this the definition of J_k is

$$(22)\quad J_k(\alpha_1{}^{l_1}, \cdots, \alpha_n{}^{l_n}) = \sum (\alpha_1\alpha_2 \cdots \alpha_n)(\alpha_1 y)^{k_1} \cdots (\alpha_n y)^{k_n} (\alpha_1 x)^{l_1-k_1-1} \cdots (\alpha_n x)^{l_n-k_n-1}$$
$$(k_1 + k_2 + \cdots + k_n = k, \qquad 0 \geq k \geq l).$$

Thus J_k has the order k in y and the order $l - k$ in x. Evidently $J_k \equiv 0$ if $k > l$.

The identity (21) is proved by repeated use of the elementary identity connecting $n + 1$ linear forms. It has the order k in y, the order $l + m - k$ in x, and it is linear in the coefficients of the given form $(\alpha_i y)^{l_i}$ $(i = 1, \cdots, n)$ and of $(\delta y)^m$. Naturally both in (21) and in (22) symbolic terms with negative exponents are not used.

Let, in (21), $k = m > l$. Then the first term of the identity does not appear since, as noted above, $J_k \equiv 0$. The order in x of the identity is l. We take then the apolarity invariant of the terms of the identity in x with the unique apolar form, $A = (a\xi)^l$, the ξ's being contragredient to the x's. Then the remaining terms on the left of (21) disappear, since $(\alpha_i x)^{l_i}$ is apolar to $(a\xi)^l$. The first term on the right of (21) is J in variables x and, according to (20) the apolarity invariant of this with $(a\xi)^l$ is R. Transposing the remaining terms on the right of (21), we have the theorem:

(23) *If $(\delta y)^m$ is an arbitrary form of order $m > l$, then $R \cdot (\delta y)^m$ can be expressed in terms of $(\alpha_i y)^{l_i}$ $(= 1, \cdots, n)$ as in* (1) *with*

$$(\beta_i y)^{m-l_i} = (-1)^{n-i}[J_{m-l_i}(\alpha_1{}^{l_1}, \cdots, \alpha_{i-1}{}^{l_{i-1}}, \alpha_{i+1}{}^{l_{i+1}}, \cdots, \alpha_n{}^{l_n}, \delta^m), (a\xi)^l],$$

the bracket being the apolarity invariant with respect to x, ξ.

This fundamental expression for $R \cdot (\delta y)^m$ $(m > l)$ has all the elasticity which is desirable in that as $(\delta y)^m$ varies in a pencil, the coefficients $(\beta_i y)^{m-l_i}$ also vary in projective pencils. The coefficients have the desired covariant character but the geometric conditions on them which limit them to the above specific forms are by no means apparent. In the preceding memoir unique coefficients were also obtained by imposing relatively simple geometric conditions but this advantage entailed the introduction of extraneous parameters.

4. Extension of the expression (1) to cover cases $m \leq l$. We again take $k = m$, and again take the apolarity invariant of the identity (21) in x with respect to $(a\xi)^l$. The effect on the terms on the right of (21) is the same as before, as well as the effect on those terms on the left of (21) which are in $\sum_{i=1}^{i=n}$. However, $J_m(\alpha_1{}^{l_1}, \cdots, \alpha_n{}^{l_n})$ is no longer identically zero. But the

apolarity invariant of $J_m \cdot (\delta x)^m$ and $(\alpha\xi)^l$ is the apolarity invariant of J_m itself and the polar $(a\delta)^m(\alpha\xi)^{l-m}$ times $[l]_{l-m} \cdot m\,!$. Hence

(24) *If $(\delta y)^m$ is an arbitrary form of order $m \leq l$, then $R \cdot (\delta y)^m$ can be expressed precisely as in* (23) *except for an additive term,*

$$[l]_{l-m} \cdot m\,! \,[J_m(\alpha_1{}^{l_1}, \cdots, \alpha_n{}^{l_n}), (\alpha\delta)^m(a\xi)^{l-m}].$$

For example, if $m = l$, the extension of the given module by the jacobian of the given forms, gives enough freedom to express all forms of order l.

The unique apolar form, when $R \neq 0$, may be defined as the *fundamental combinant* of the module determined by $(\alpha_1 y)^{l_1}, \cdots, (\alpha_n y)^{l_n}$. Indeed, with $l_1 \leq l_2 \leq \cdots \leq l_n$, and with $(a\eta)^l$ given, $(\alpha\, y)^{l_1}$ is the form of lowest order apolar to $(a\eta)^l$ unless $l_1 = l_2 = \cdots = l_k$ in which case any k linearly independent forms of order l_1 apolar to $(a\eta)^l$ constitute the first k forms of the module. Then $(\alpha_{k+1}y)^{l_{k+1}}$ is one form, or one of j linearly independent forms, of order l_{k+1} both apolar to $(a\eta)^l$ and also not found in the module determined by $(\alpha_1 y)^{l_1}, \cdots, (\alpha_k y)^{l_k}$. Proceeding in this way the module may be obtained from the apolar forms of $(a\eta)^l$ alone.

To all of the above there is one exception even when $R \neq 0$. This is the case where $l_n > l$ or $0 > l_1 + \cdots + l_{n-1} - n$ so that $l_1 = l_2 = \cdots = l_{n-1} = 1$. Then the apolarity of the last form with $(a\eta)^l$ presents no conditions and there is no significant apolar form of class l when $R \neq 0$. This exception occurs first in the binary case $l_1, l_2 = 1, l_2$ and it persists as new variables and linear forms are added.

As an example of the transition from a module to its unique apolar form and vice versa consider the ternary module determined by a ternary quadratic, cubic, and quartic, say $(\alpha y)^2$, $(\beta y)^3$, $(\gamma y)^4$, each generic. Then, if $(p_1\eta), \cdots, (p_6\eta)$ are the six points common to $(\alpha y)^2$, $(\beta y)^3$, according to (15) and (17), we have

$$(25, \text{a}) \qquad A = \lambda_1(p_1\eta)^6 + \lambda_2(p_2\eta)^6 + \cdots + \lambda_6(p_6\eta)^6,$$

$$(25, \text{b}) \qquad 0 \equiv (p_1\eta)^2 + (p_2\eta)^2 + \cdots + (p_6\eta)^2.$$

With A as given in (25, a), the conic $(\alpha y)^2$ is the unique conic on $p_1, \cdots, p_6$ whose existence is a consequence of (25, b). The cubic $(\beta y)^3$ is any cubic on $p_1, \cdots, p_6$ which does not contain the conic as a part. The quartic $(\gamma y)^4$ is any quartic apolar to A which is not in the module determined by $(\alpha y)^2$, $(\beta y)^3$. The module depends projectively on the 11 constants involved in the choice of the six points on a conic and the 5 additional constants which arise from the ratios of the λ's. Thus A itself is a quite special sextic of class six subject to 11 conditions.

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ASYMPTOTIC INTEGRATION CONSTANTS.*

By AUREL WINTNER.

Let x and f be vectors with a common number of real components, x_i and f_i $(i = 1, \cdots, k)$, and let $f = f(x, t)$ be defined and continuous at every point (x, t) of the product space of the whole Euclidean x-space and of the half-line $0 \leqq t < \infty$. Suppose further that, if the sign of absolute value refers to Euclidean length, the inequality

$$|f(x,t)| \leqq \lambda(t)\phi(|x|) \tag{1}$$

holds for a positive and, for instance, continuous function $\lambda(t), 0 \leqq t < \infty$, satisfying

$$\int_0^\infty \lambda(t)dt < \infty \tag{2}$$

and for a function $\phi(r)$ which is defined on the closed half-line $0 \leqq r < \infty$, is positive and continuous on the open half-line $0 < r < \infty$ and satisfies

$$\int_1^\infty (dr)/\phi(r) = \infty. \tag{3}$$

Let the set of all these conditions be denoted by (*).

Clearly, (*) is a restriction of $f(x, t)$ itself (for large $|x| + |t|$), rather than one of the Lipschitz-Osgood difference

$$f(x^*, t) - f(x^{**}, t) \tag{4}$$

(for small $|x^* - x^{**}|$). Correspondingly, the uniqueness of the solution $x = x(t)$ of

$$x' = f(x, t), \qquad x(0) = x^0, \tag{5}$$

where the initial vector x^0 is given and x' denotes dx/dt, is not ensured by (*). Nevertheless, it is easy to prove the following lemma (cf. [2]):

(i) *If $f(x, t)$ satisfies (*), then every solution $x = x(t)$ of (5), where x^0 is given arbitrarily, exists for $0 < t < \infty$ and tends to a finite limit vector, $x(\infty)$, as $t \to \infty$.*

* Received April 23, 1946.

Let (?) denote the set of those points of the x-space which represent limiting positions, $x(\infty)$, of solutions, $x = x(t)$, of $x' = f(x, t)$. Then (i) does not claim that (?) is the whole x-space, i. e., that, *instead of the initial vector, $x(0)$, one can assign the final vector, $x(\infty)$, as an arbitrary integration constant.* In fact, the chances for the truth of this converse of (i) seem to be lowered by the circumstance that the assumption, (*), of (i) does not ensure the uniqueness of the solution, $x = x(t)$, of (5). But it turns out that this impression is misleading, since (*) alone implies that (?) is the whole x-space:

(ii) *If $f(x, t)$ satisfies (*), then*

$$x' = f(x, t), \qquad x(\infty) = {}^0x, \tag{6}$$

where ${}^0x$ is given arbitrarily, has a solution $x = x(t)$, $0 \leqq t < \infty$.

Clearly, the assumptions imposed by (*) on $f(x, t)$ remain satisfied if $f(x, t)$ is replaced by $f(x - a, t)$, where a is an arbitrary constant vector. Hence, it is sufficient to prove (ii) for the case ${}^0x = 0$, where 0 denotes the zero vector. In other words, the assertion of (ii) is that, if $f(x, t)$ satisfies (*), then

$$x(t) \to 0 \text{ as } t \to \infty \tag{7}$$

must hold for some solution $x = x(t)$, $0 \leqq t < \infty$, of $x' = f(x, t)$.

In order to establish the existence of such a solution of $x' = f(x, t)$, choose an arbitrary $t_0 \geqq 0$ and consider a solution $x = x(t)$ satisfying

$$x(t_0) = 0. \tag{8}$$

It is clear from the first of the assertions of (i) that this solution must exist for $0 \leqq t < \infty$. In view of (1), it satisfies the inequality

$$\int_u^v |\, dx(t)\,| / \phi(\,|\, x(t)\,|\,) \leqq \int_u^v \lambda(t)\, dt \tag{9}$$

on every interval $u \leqq t \leqq v$. But the assumptions imposed by (*) on $\phi(r)$, where $0 \leqq r < \infty$, are (1), (3) and the condition that $\phi(r)$ be positive and

continuous when $r \neq 0$, and all these assumptions remain satisfied if $\phi(r)$ is replaced by the function which is $\max'(\phi(1), \phi(r))$ or $\phi(r)$ according as $0 \leqq r \leqq 1$ or $r > 1$. Hence, it can be assumed that $\phi(r)$ is positive and continuous at $r = 0$ also. Then, since $x(t)$ is differentiable, $1/\phi(|x(t)|)$ is continuous throughout and cannot, therefore, omit any value between values attained. Consequently, from (9) and (8),

$$\int_0^R (dr)/\phi(r) \leqq \int_0^\infty \lambda(t)\,dt, \text{ where } R = \underset{0 \leqq t < \infty}{\text{l. u. b.}} |x(t)|, \tag{10}$$

the lower limit of integration on the left of (10) being

$$0 = x(t_0) = \underset{0 \leqq t < \infty}{\text{g. l. b.}} |x(t))|. \tag{11}$$

The relations (10), (3), (2) imply the existence of a constant C having the property that

$$|x(t)| < C \text{ for } 0 \leqq t < \infty, \tag{12}$$

where $x(t)$ is any solution satisfying (8) for some t_0, and C is independent of the choice of this solution (including the choice of t_0). However, since $\phi(r)$ has a positive minimum on every closed, bounded interval $0 \leqq r \leqq C$, it is seen from (12) and (9) that

$$\int_u^v |dx(t)| \leqq c \int_u^v \lambda(t)\,dt, \tag{13}$$

where $c = c(C)$ is independent of the choice of u, v in (13) and of that of t_0 in (8).

Let $t_0 = n(= 1, 2, \cdots)$, and let $x^n(t)$, where $0 \leqq t < \infty$, be a corresponding solution of $x' = f(x, t)$. Then, from (13) and (2),

$$\int_0^\infty |dx^n(t)| < \text{const.} \tag{14}$$

In view of $x^n(n) = 0$, this is a refinement of

$$|x^n(t)| < C; \tag{15}$$

cf. (8). In addition, (13) and (2) imply that, for every non-negative T,

$$\int_T^\infty |\, dx^n(t)\,| < \epsilon(T), \tag{16}$$

where $\epsilon = \epsilon(T)$ is independent of n and tends to 0 as $T \to \infty$. Finally, (16) implies, besides the existence of a finite limit vector $x^n(\infty)$, the inequality

$$|\, x^n(T) - x^n(\infty)\,| < \epsilon(T).$$

This means that

$$x^n(t) \to x^n(\infty) \text{ as } t \to \infty \text{ holds uniformly in } n. \tag{17}$$

Since $f(x, t)$ is continuous on the product space of the whole x-space and of the half-line $0 \leqq t < \infty$, it is seen from (15) and (1) that the sequence $f(x^1(t), t), f(x^2(t), t), \cdots$ is uniformly bounded on every bounded interval of this half-line. But the n-th element of the sequence is the derivative of $x^n(t)$. Hence, the sequence $x^1(t), x^2(t), \cdots$ is equicontinuous on every bounded interval of the half-line. Since it is uniformly bounded, it contains a subsequence which is uniformly convergent on every bounded interval. In view of (15) and of the existence of every $x^n(\infty)$, this subsequence of the functions $x^1(t), x^2(t), \cdots$ can be chosen so as to make the corresponding subsequence of the limits $x^1(\infty), x^2(\infty), \cdots$ a convergent sequence of constants. Then it is seen from (17) that the resulting subsequence of $x^1(t), x^2(t), \cdots$ is uniformly convergent on the half-line $0 \leqq t < \infty$.

Let $x(t)$, where $0 \leqq t < \infty$, denote the limit function of such a subsequence. Then $x = x(t)$ is a solution of $x' = f(x, t)$, since every $x = x^n(t)$ is. Furthermore, $x(\infty)$ exists, by (i) or (14). But if the n-th element of the subsequence is denoted by $x^m(t)$, where $m = m_n$, then

$$\text{l. u. b.}_{0 \leqq t < \infty} |\, x^m(t) - x(t)\,| \to 0 \text{ as } n \to \infty.$$

In particular, since $x^m(\infty)$ and $x(\infty)$ exist,

$$x^m(\infty) \to x(\infty) \text{ as } n \to \infty.$$

Finally, since $x^n(n) = 0$,

$$x^m(m) = 0 \text{ where } m \to \infty \text{ as } n \to \infty.$$

In view of (17), the last three formula lines imply that the value of the constant $x(\infty)$ cannot be distinct from 0. Since this means that the solution $x(t)$ constructed satisfies (7), the proof of (ii) is complete.

Appendix.

Condition (*) has a dual which, when substituted for (*), leaves the assertion of (i) unaltered but transforms the situation expressed by (ii) into the complete opposite of that situation, in the following sense: The limiting vector $x(\infty)$, instead of being connected with the initial vector $x(0)$ in such a way as to be capable of *every* position in the x-space, becomes a position which is independent of $x(0)$; so that all solution paths of $x' = f(x, t)$ become "confluent" at a single point of the x-space, as $t \to \infty$.

Let (* bis) denote the set of conditions which results from the set (*) if (1) is replaced by

(1 bis) $$x \cdot f(x, t) \leqq -\lambda(t)\phi(x \cdot x)$$

and (2) by

(2 bis) $$\int_0^\infty \lambda(t)\,dt = \infty, \qquad (\lambda \geqq 0)$$

finally (3) by

(3 bis) $$\int_\epsilon^R (dr)/\phi(r) < \infty \text{ if } 0 < \epsilon < R < \infty \qquad (\phi \geqq 0)$$

(it is understood that $x \cdot y$ in (1 bis) denotes the scalar product $x_1 y_1 + \cdots + x_k y_k$). It will be shown that, corresponding to the duals (1 bis), (2 bis), (3 bis) of (1), (2), (3), the "confluent" counterpart of (i) and (ii) can be formulated as follows:

If $f(x, t)$ satisfies (* bis), *then every solution $x = x(t)$ of* (5), *where x^0 is arbitrary, exists for $0 \leqq t < \infty$ and satisfies* (7).

First, if $r = r(t)$ denotes $|x|^2$, where $x = x(t)$, then $x' = f(x, t)$ implies that $\frac{1}{2}r'$ is the scalar product $x \cdot f(x, t)$. Hence, from (1 bis),

$$r'(t) \leqq -2\lambda(t)\phi(r(t)),$$

and so, since $\lambda \geqq 0$ and $\phi \geqq 0$, the derivative of $|x(t)|^2$ is non-positive throughout. Consequently, if t increases from $t = 0$ onward, $|x(t)|$ cannot increase. Since this precludes the existence of a $t^0 > 0$ satisfying $|x(t)| \to \infty$ as $t \to t^0 - 0$, and since $f(x, t)$ is supposed to be defined and continuous on the product space of the whole x-space and of the half-line $0 \leqq t < \infty$, it follows from a general fact concerning systems $x' = f$ (cf. [1], p. 177), that every solution $x = x(t)$ of (5) exists for $0 \leqq t < \infty$.

Since the derivative of $r(t) = |x(t)|^2$ is non-positive, it is also seen that there exists a finite limit $r(\infty) \geqq 0$. Hence, in order to complete the proof of the last italicized statement, all that remains to be shown is that $r(\infty) > 0$ is impossible. But this can be concluded from the assumptions which have not been used thus far. In fact, if $r(t)$ does not vanish for all t from a certain $t = t^*$ onward, say for $t \geqq 0$, then, by the last formula line, the inequality

$$\int_0^T |dr(t)|/\phi(r(t)) \geqq \int_0^T \lambda(t)dt$$

holds for every $T > 0$, since $r(t)$ is monotone and $\phi \geqq 0$, $\lambda \geqq 0$. Consequently, from (2 bis),

$$\int_0^\infty |dr(t)|/\phi(r(t)) = \infty,$$

and so, since $r(t)$ is a continuous, non-increasing function,

$$\int_{r(\infty)}^{r(0)} (dr)/\phi(r) = \infty, \text{ where } 0 \leqq r(\infty) \leqq r(0) < \infty.$$

It follows therefore from (3 bis) that $r(\infty) > 0$ is impossible.

The "o-theorem" just proved has an "O-variant," which in the linear case can be formulated as follows:

If $A(t)$, where $0 \leqq t < \infty$, is a matrix of n times n real-valued continuous functions satisfying the unilateral restriction

$$\limsup_{t\to\infty} \int_0^t \max_{|y|=1} (y \cdot A(s)y)ds < \infty,$$

then every solution vector $x = x(t)$ of $x' = A(t)x$ is bounded as $t \to \infty$.

The restriction is unilateral, since the integral, the upper limit of which is required to be distinct from $+\infty$, is allowed to have the lower limit $-\infty$. In fact, the function integrated is the maximum, rather than the maximum of the absolute value, attained on the real unit sphere $y_1^2 + \cdots + y_n^2 = 1$ by the form which results if the two sets of variables are indentified in the bilinear form belonging to the matrix. In other words, the function integrated

is the greatest, rather than the absolutely greatest, characteristic number of the arithmetical mean of the matrix and its transposed matrix.

The proof follows if $x' = A(t)x$ is multiplied by $x = x(t)$. Since $x \cdot x' = |x|\,|x|'$, this shows that $|x(t)|\,|x(t)|'$ does not exceed $|x(t)|^2$ times the maximum of $y \cdot A(t)y$ on the sphere $|y| = 1$. But $|x(t)|$ cannot vanish at any $t = t_0$ unless it vanishes identically, since $x(t) \equiv 0$ is one, hence the only, solution of $x' = A(t)x$ and of the initial condition $x(t_0) = 0$. Hence, if the trivial solution $x(t) \equiv 0$ is excluded, division by $|x(t)|^2$ is allowed, and so the preceding estimate means that $(\log|x(t)|)'$ does not exceed the maximum of $y \cdot A(t)y$ on the sphere $|y| = 1$. It follows, therefore, from the last formula line that $\log|x(t)|$ *is bounded from above,* which proves that $|x(t)|$ is bounded.

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THE MEASURE THEORETIC APPROACH TO DENSITY.*

By R. Creighton Buck.

1. Introduction. The density of a set of integers $A = \{a_n\}$ is usually defined as $D(A) = \lim n/a_n$. This function has some of the properties of a finitely additive measure on the countable space composed of the positive integers, although it is true that sets A and B may have a density while $A \cup B$ does not. However, it is clear that generalized definitions of density can be given which apply to all sets of integers, and which are in fact true measures. [1, 231] [4].

The present paper is largely devoted to an analysis of the measure defined on the set of positive integers by applying the Carathéodory extension to a simple basic measure; connection with the theory of Jordan content is very close. This also provides a simple model for classical measure theory; since points are to have zero measure, while the space on which the measure is defined is only countable, we must require only finite additivity. It is also clear that in studying the set of integers, we are studying any countable discrete space, for such a space can be mapped onto I.

In Section **2**, we construct the measure μ and the class $\mathfrak{D}_\mu$ of measurable sets. In Section **3**, we prove measurability of certain special sets using number theoretic methods. We discuss in Section **4** certain questions related to sequences of sets and prove that the range of μ is precisely the closed unit interval. Section **5** is devoted to an analysis of what we have called 'quasi-progressions.' It is proved that if α is irrational, the set of integers of the form $[\alpha n + \beta]$ intersects every arithmetic progression in an infinite set. In a sense, this is dual to the fact that if α is irrational, the fractional parts of $\alpha n + \beta$ are everywhere dense in the unit interval. In Section **6**, we discuss a number of properties of ordinary density, and in the following section examine certain of its generalizations, using results dealing with regular summability methods. The concluding section deals with the customary dyadic mapping and questions of the relative measure of classes of measurable or densable sets. Several problems are left open.

2. Measure density. Let I denote the set of all positive integers. On the class of arithmetic progressions, we have what we may call a natural

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definition of density. If A is $\{an+b\}$ we define its density to be $\Delta(A) = 1/a$; this we take as a starting point. A measure density on I will be a measure defined for a class of subsets of I which is finitely additive and under which a progression A has measure $\Delta(A)$. We shall require that a point have measure zero, so that the same is true for any finite set; it is then clear that altering a finite number of points of a measurable set will not change its measure. A dot placed above the symbol for a relation will be used to indicate that the relation holds modulo the class of finite sets; thus, $A \dot{\subset} B$ means that if a finite set is deleted from both sets, we will have $A \subset B$, while $A \doteq 0$ means that A itself is finite.

DEFINITION A. *$\mathfrak{D}_0$ is the class of all sets $A \subset I$ which are finite unions of arithmetic progressions, or which differ from these by finite sets.*

This class has the following properties:

(A1) If $A \in \mathfrak{D}_0$, then $A' \in \mathfrak{D}_0$, where A' is the complement of A.

(A2) If $A \in \mathfrak{D}_0$, $B \in \mathfrak{D}_0$, then $A \cup B$ and $A \cap B$ belong to $\mathfrak{D}_0$.

(A3) If $A \in \mathfrak{D}_0$ and $A \doteq B$, then $B \in \mathfrak{D}_0$.

The function Δ may now be extended to $\mathfrak{D}_0$.

DEFINITION B. *If A is a progression $\{an+b\}$, then $\Delta(A) = 1/a$; if A is the union of the disjoint progressions $A_1, A_2, \cdots A_r$, then $\Delta(A) = \Delta(A_1) + \Delta(A_2) + \cdots + \Delta(A_r)$; if $\Delta(A)$ is defined and $A \doteq B$, then $\Delta(B) = \Delta(A)$.*

The measure function Δ clearly has the following properties.

(B1) If A and B belong to $\mathfrak{D}_0$ and $A \dot{\subset} B$, then $\Delta(A) \leq \Delta(B)$.

(B2) If A and B belong to $\mathfrak{D}_0$ and $A \cap B \doteq 0$ then

$$\Delta(A \cup B) = \Delta(A) + \Delta(B).$$

(B3) If A and B belong to $\mathfrak{D}_0$ then

$$\Delta(A \cup B) + \Delta(A \cap B) = \Delta(A) + \Delta(B).$$

This last property is obviously implied by (B2).

We now define an outer measure on I in terms of Δ. The statements that follow are left unproved since they follow as in classical measure theory, or in that of Jordan content. [6]

DEFINITION C. *If* $S \subset I$, $\mu(S) = \inf \Delta(A)$ *for* $A \dot{\supset} S$ *and* $A \in \mathfrak{D}_0$.

(C1) If $S_1 \dot{\subset} S_2$ then $\mu(S_1) \leq \mu(S_2)$.

(C2) $\mu(S_1 \cup S_2) \leq \mu(S_1) + \mu(S_2)$.

(C3) If $A \in \mathfrak{D}_0$ then $\mu(A) = \Delta(A)$.

DEFINITION D. $\mathfrak{D}_\mu$ *is the class of all sets* S *for which*

$$\mu(S) + \mu(S') = 1.$$

This class is the Carathéodory extension of $\mathfrak{D}_0$ since the definition above is equivalent to either of the following:

(i) S belongs to $\mathfrak{D}_\mu$ if for any set X,

$$\mu(X) = \mu(X \cap S) + \mu(X \cap S').$$

(ii) S beongs to $\mathfrak{D}_\mu$ if, given $\epsilon > 0$, there exist sets A and B in $\mathfrak{D}_0$ with $A \dot{\subset} S \dot{\subset} B$ and $\Delta(B) - \Delta(A) = \Delta(B - A) < \epsilon$.

In this class, μ is a measure density with the usual properties.

(D1) If $S \in \mathfrak{D}_\mu$, then $S' \in \mathfrak{D}_\mu$.

(D2) If S_1 and S_2 belong to $\mathfrak{D}_\mu$, then so do $S_1 \cup S_2$ and $S_1 \cap S_2$.

(D3) If S_1 and S_2 are any two sets of $\mathfrak{D}_\mu$, then

$$\mu(S_1 \cup S_2) + \mu(S_1 \cap S_2) = \mu(S_1) + \mu(S_2).$$

Since μ is obtained as the Carathéodory extension of Δ it is in one sense the most natural measure density in I. In the next section we shall discuss the class $\mathfrak{D}_\mu$ and in particular show that it *properly* contains $\mathfrak{D}_0$.

3. Special sets. An immediate consequence of the definition of $\mathfrak{D}_0$ is that if $A \in \mathfrak{D}_0$ and $\Delta(A) = 0$, then $A \doteq 0$. We begin by showing that $\mathfrak{D}_\mu$ contains infinite sets of measure zero.

THEOREM 1. *Let* P_0 *be a set of primes such that* $\sum_{P_0} 1/p = \infty$. *Let* S *be a set of integers having the property that if* $p \in P_0$, *no more than a finite number of integers of* S *are divisible by* p. *Then,* $\mu(S) = 0$.

For example, P_0 can be taken as the set of all primes, or all the primes in an arithmetic progression $\{an + b\}$ where $(a, b) = 1$, or the primes p_{λ_n} where $\lambda_n \sim n \log\log n$. In particular, S can be any coprime set such as the

set of all primes or the set $\{2^{2^n}+1\}$. The less inclusive P_0 is, the more inclusive S may be.

Proof. Let λ be a product of primes of P_0, and let A_k be the arithmetic progression $\{\lambda_n + k\}$ for $k = 1, 2, \cdots \lambda$. Every integer of A_k is divisible by (λ, k) and this in turn is either 1, or itself a product of primes of P_0. Consider the set $S \cap A_k$ in case $(\lambda, k) \neq 1$. Each element of this set is divisible by at least one prime of P_0 which is also a divisor of λ. By hypothesis, only a finite number of terms of S are divisible by any one prime of P_0, and hence by any of the finite collection of primes dividing λ. We conclude that the set $S \cap A_k$ is finite. Writing $I = \underset{k\leq\lambda}{U} A_k$, we have $S = \underset{k\leq\lambda}{U} (S \cap A_k) \doteq \underset{k\leq\lambda}{U'} (S \cap A_k)$, where the dash indicates that the union is to be taken only for k with $(\lambda, k) = 1$. Hence we have $S \doteq \underset{k\leq\lambda}{U'} A_k$ and $\mu(S) \leq \underset{k\leq\lambda}{\sum'} \Delta(A_k)$ $= \phi(\lambda)/\lambda$ where, as usual, $\phi(m)$ is Euler's phi function. Let us now choose λ as $\underset{p_0\leq m}{\prod} p_0$ where the subscript indicates that we are considering only primes belonging to P_0. Since $\phi(\lambda) = \lambda \underset{p_0\leq m}{\prod} (1 - 1/p_0) \leq \lambda \exp\{-\underset{p_0\leq m}{\sum} 1/p_0\}$ we have $\mu(S) \leq \exp\{-\underset{p_0\leq m}{\sum} 1/p_0\}$; using our assumed property of the set P_0, and letting m increase, we conclude that $\mu(S) = 0$.

Using similar methods we can show that other special sets possess a μ measure. For example, let S be the set $\{n^2\}$. Modulo λ, this set collapses into a finite collection of distinct numbers $r_1, r_2, \cdots r_\alpha$ which we call the *complete* set of quadratic residues modulo λ. This differs from the customary definition in admitting residues not relatively prime to the modulus. We denote the number of these residues by $w(\lambda) = \alpha$. It is clear that

$$S \subset \{\lambda n + r_1\} \cup \{\lambda n + r_2\} \cup \cdots \cup \{\lambda n + r_\alpha\}$$

and that therefore $\mu(S) \leq w(\lambda)/\lambda$ for all λ. We shall show that lim inf $w(\lambda)/\lambda = 0$, and thus that $\mu(S) = 0$. For this, it would be sufficient to know that $w(n)$ is multiplicative, that if p is any prime $w(p) = (p+1)/2$, and that $\underset{1}{\overset{n}{\prod}} (1 + 1/p_k) = o(p_n)$. However, the following more complete result is of interest in itself[1]

THEOREM 2. *The function $w(n)$ has the following properties which suffice to define it completely:*

[1] A partial solution appears in Uspensky, *Elementary Theory of Numbers*, p. 324.

(i) *If* $(a, b) = 1$, *then* $w(ab) = w(a)w(b)$.

(ii) $w(p^n) = p^{n+1}/(2p+2) + \begin{cases} (p+2)/(2p+2) \text{ for } n \text{ even} \\ (2p+1)/(2p+2) \text{ for } n \text{ odd.} \end{cases}$

(iii) $w(2^n) = 2^{n-1}/3 + \begin{cases} 4/3 \text{ if } n \text{ is even} \\ 5/3 \text{ if } n \text{ is odd.} \end{cases}$

Proof. (i) Let $\alpha = w(a)$, $\beta = w(b)$, and let $r_1, r_2, \cdots r_\alpha$, $s_1, s_2, \cdots s_\beta$ be the complete sets of quadratic residues modulo a and b respectively. Consider the sets of integers $\{r_i + ax\}$ and $\{s_j + by\}$ where $x = 0, 1, \cdots b-1$, $y = 0, 1, \cdots a-1$, $i = 1, 2, \cdots \alpha$ and $j = 1, 2, \cdots \beta$. The numbers in these sets are the quadratic residues modulo a and b respectively that lie between 0 and ab. Their common part is exactly the complete residue set modulo ab. Since $(a, b) = 1$ we can find unique x and y corresponding to any pair i, j, such that $ax - by = s_j - r_i$. For each pair i, j, there is then a quadratic residue modulo ab, and these are all distinct; their number is then $\alpha\beta = w(a)w(b) = w(ab)$.

(ii) Since $x^2 \equiv (-x)^2$, we have $w(p) = 1 + (p-1)/2 = p^2/(2p+2) = (2p+1)/(2p+2)$, and the formula holds for $n = 1$. To compute $w(p^2)$, we observe that the $\phi(p^2)/2$ integers $1, 2, \cdots p-1$, $p+1, \cdots (p^2-1)/2$, omitting all multiples of p, have distinct squares modulo p^2. The remaining integers, $p, 2p, \cdots (p-1)p$, p^2 have the same square modulo p^2, namely zero. Thus, $w(p^2) = 1 + p(p-1)/2 = p^3/(2p+2) + (p+2)/(2p+2)$, and the formula holds for $n = 2$. For the general case, suppose that x and y are prime to p while $x^2 \equiv y^2 \pmod{p^n}$. Then $(x+y)(x-y) \equiv 0$ and since both factors cannot be divisible by p unless x and y are, we infer that $x \equiv \pm y \pmod{p^n}$. Thus of the $\phi(p^n)$ integers between 1 and p^n not divisible by p, exactly half give rise to incongruent squares. Suppose now that $x = px'$, $y = py'$ while $x^2 \equiv y^2 \pmod{p^n}$. This implies that $x'^2 \equiv y'^2 \pmod{p^{n-2}}$ and the number of incongruent squares obtained is $w(p^{n-2})$. Combining these, we see that $w(p^n) = \phi(p^n)/2 + w(p^{n-2})$; solving this difference equation with initial conditions corresponding to the values of $w(p)$ and $w(p^2)$, we obtain formula (ii).

(iii) The proof of this is quite similar. We readily find that $w(2) = w(4) = 2$. In the general case, since $x^2 \equiv (-x)^2 \equiv (x + 2^{n-1})^2 \pmod{2^n}$, we need only consider the squares $1^2, 2^2, \cdots (2^{n-2})^2$. By an argument similar to that used in the proof of (ii), we find that $w(2^n) = \phi(2^n)/4 + w(2^{n-2})$; solving this, we arrive at formula (iii).

THEOREM 3. *Lim sup* $w(n)/n = 1/2$ *and* $\liminf w(n)/n = 0$.

From the preceeding theorem, we have $w(p^n)/p^n \leq 2^{-1} + p^{-n}$ and $w(2^n)/2^n \leq 6^{-1} + 2^{1-n}$. Hence, if $n = 2^\beta \prod_1^m p_{r_i}^{\beta_i}$ with $\beta_i \geq 1$

$$\begin{aligned} w(n)/n &\leq (6^{-1} + 2^{1-\beta})2^{-m} \prod_1^m (1 + 2p_{r_i}^{-\beta_i}) \\ &\leq (6^{-1} + 2^{1-\beta})2^{-m} \exp 2 \sum_1^m p_{r_i}^{-\beta_i} \\ &\leq (6^{-1} + 2^{1-\beta})2^{-m} m^2 e^2 \end{aligned}$$

since $\sum_1^m p_{r_i}^{-\beta_i} \leq \sum_1^m p_i^{-1} \leq \sum_1^m k^{-1} \leq 1 + \log m$. If n tends to infinity along a sequence of integers in such a manner that m, the number of distinct prime factors, also approaches infinity, then the uniform estimate obtained above shows that $w(n)/n$ will approach zero. If n tends to infinity through a sequence of values for which m is bounded, then we will obtain the maximum values of $w(n)/n$ by considering only $n = p^m$. We have $w(p^m)/p^m = p/(2p+2) + O(1)p^{-m}$; letting m tend to infinity gives us $p/(2p+2)$, and as p increases, we obtain $\limsup w(n)/n = 1/2$. For the limit inferior, choose $n = \prod_{p \leq m} p^2$; since $w(p^2) = (p^2 - p + 2)/2$, we see that $w(p^2)/p^2$ is not greater than $1/2$ for $p \geq 3$, and therefore $w(n)/n = \prod_{p \leq m} w(p^2)/p^2 \leq 2^{-\pi(m)}$ which approaches zero as m increases.

COROLLARY. *If* S *is the set* $\{n^2\}$, *then* $\mu(S) = 0$.

We have constructed infinite sets having μ measure zero. Since the only set having Δ measure zero are finite sets, we have shown that the inclusion $\mathfrak{D}_0 \subset \mathfrak{D}_\mu$ is strict. From these null sets of $\mathfrak{D}_\mu$ we can build other measurable sets; thus, for example, if $A \in \mathfrak{D}_0$ and $Z \in \mathfrak{D}_\mu$ with $\mu(Z) = 0$, then $A \cup Z$ and $A - Z$ are in $\mathfrak{D}_\mu$ and have measure $\Delta(A)$. We observe that if Z is infinite both of the sets $A \cup Z$ and $A - Z$ cannot belong to $\mathfrak{D}_0$, since $Z = (A \cup Z) \cap (A - Z)'$. For comparison we recall that the general Lebesgue measurable set may be expressed either as a G_δ minus a set of measure zero, or as an F_σ plus a set of measure zero. [6, p. 80.] We now show that in the μ measure, no set of $\mathfrak{D}_\mu$ can be expressed in both forms.

THEOREM 4. *If* $S = A \cup Z = B - W$, *where* A *and* B *belong to* $\mathfrak{D}_0$ *and* $\mu(Z) = \mu(W) = 0$, *then* S *is also in* $\mathfrak{D}_0$.

First, we have $\mu(S) = \Delta(A) = \Delta(B)$. We may write S as $(A \cup Z) \cap (B - W) = (A \cap B \cap W') \cup (Z \cap [B - W])$. The last term has measure

zero since it is a subset of Z, so that $\mu(S) = (A \cap B \cap W') = \Delta(A \cap B)$. Now, $\Delta(A) = \Delta(A \cap B) + \Delta(A - B)$; substituting, we see that $\Delta(A - B) = 0$ and that therefore $A - B \doteq 0$. Similarly, from the identity $\Delta(B) = \Delta(B \cap A) + \Delta(B - A)$, we conclude that $B - A \doteq 0$. Combining these we have $A \doteq B$, and since $A \subset S \subset B$, $S \doteq A$ and S belongs to $\mathfrak{D}_0$.

A natural question arises: are all the sets of $\mathfrak{D}_\mu$ obtainable in this fashion, as sets of the form $A \cup Z$ or $A - Z$ where $A \in \mathfrak{D}_0$ and $\mu(Z) = 0$? The answer lies in the fact that since every set of $\mathfrak{D}_0$ is, except for a finite set, the union of a finite number of progressions, the measure of any set having either of these forms must be a rational number. In the next section, we construct sets of $\mathfrak{D}_\mu$ having any given irrational number for its measure.

4. Limit sets. We begin by considering sequences of sets. One might be led by analogy with ordinary measure to conjecture that if $A_1 \subset A_2 \subset \cdots$ is a sequence of sets of $\mathfrak{D}_0$, then the limit set $S = \lim A_n$ must belong to $\mathfrak{D}_\mu$ and have measure $\mu(S) = \lim \Delta(A_n)$. A simple example shows that this is false; take $A_n = \{1, 2, \cdots n\}$. These sets increase, each has measure zero, but $\lim A_n = I$. However, we observe in this example that if $S = \{1\}$, then $A_n \dot{\subset} S$ for all n and $\mu(S) = \lim \Delta(A_n)$. We might therefore be led to conjecture instead that if $A_1 \subset A_2 \subset \cdots$, then there exists a set S such that $A_n \dot{\subset} S$ for all n and $\mu(S) = \lim \Delta(A_n)$. If this were true, the task of constructing a set with preassigned measure would be very simple. That this conjecture too is false is shown by the following theorem.

THEOREM 5. *There exist sets $A_1 \subset A_2 \subset \cdots$ of $\mathfrak{D}_0$ such that* $\lim \Delta(A_n) = 1/3$ *while if $A_n \dot{\subset} S$ for all n, then S has outer measure $\mu(S) = 1$.*

We shall prove that if $A_n \dot{\subset} S$ for all n, then S must have an infinite number of terms in common with any arithmetic progression. This will imply that $\mu(S) = 1$. For, suppose that $B \in \mathfrak{D}_0$ and that $S \dot{\subset} B$ with $\Delta(B) < 1$. Then, B' is infinite and there is a progression E such that $E \dot{\subset} B'$; consequently $S \cap E \dot{\subset} S \cap B' \dot{\subset} B \cap B' = 0$ and $S \cap E \doteq 0$, contradicting the property of S mentioned above.

The sets A_n will be defined as $U_1{}^n C_m$ where C_m is the progression $\{4^m N + r_m\}$. We proceed to select the integers r_m. Arrange all arithmetic progressions into a countable sequence $E_1, E_2, \cdots$. Choose $r_1 = 1$ giving us $C_1 = A_1 = \{4N + 1\}$. Let E_{λ_1} be the first set E_i not a subset of A_1 and choose r_2 as any integer in $E_{\lambda_1} - A_1$. We see that $C_1 \cap C_2 = 0$ while $A_2 \cap E_i$ is infinite for each $i \leq \lambda_1$. Having chosen $r_1, r_2, \cdots r_n$, we select r_{n+1} as any integer in $E_{\lambda_n} - A_n$ where E_{λ_n} is the first set E_i following $E_{\lambda_{n-1}}$ that is not a

subset of A_n. The set C_{n+1} is disjoint from all previously defined sets C_m; moreover, $A_n \cap E_i$ is non-void, and hence infinite, for all $i \leq \lambda_n$.

Now, suppose that $A_n \dot{\subset} S$ for all n. Let E_i be any progression and suppose that $S \cap E_i \doteq 0$. But this essentially contains the set $A_n \cap E_i$ for each n, and this is infinite for at least one value of n. Finally, since the sets C_m are pairwise disjoint, $\Delta(A_n) = \sum_1^n \Delta(C_m) = \sum_1^n 4^{-m}$ and $\lim \Delta(A_n) = 1/3$.

Taking complements, we can construct a decreasing sequence of sets A_n such that $\lim \Delta(A_n) = 2/3$, while if $S \dot{\subset} A_n$ for all n, S must have inner measure zero. We remark in passing that although the second conjectured theorem fails for the class $\mathfrak{D}_\mu$, it is however true for the usual limit density D defined on the class $\mathfrak{D}$. This is proved in Section **6**.

Following similar methods, we now proceed to construct a set S of irrational measure. We first define two collections of sets $\{A_m\}$ and $\{B_k\}$ such that $A_m \subset B_k$ for all m and k while $\lim \Delta(A_m) = \lim \Delta(B_k) = \beta$, an irrational number. Then, from these we construct a set S such that $A_m \dot{\subset} S \dot{\subset} B_k$ for all m and k. From (ii), definition D, it is clear that S belongs to $\mathfrak{D}_\mu$ and has measure β.

Our approximating sets will be defined in turn by two auxiliary sequences of sets, $\{C_i\}$ and $\{D_j\}$; we set

$$A_m = C_1 \cup C_2 \cup \cdots \cup C_m;$$
$$B_k = C_1 \cup C_2 \cup \cdots \cup C_{k-1} \cup D_k.$$

In order that $A_m \subset B_k$ for all m and k, it is sufficient to have $C_m \subset D_k$ for $k = 2, 3, \cdots m$, and for all m. We will also want the sets C_i to be disjoint. Let C_m be the arithmetic progression $\{2^{m(m+1)/2}N + \gamma_m\}$ and D_k the progression $\{2^{k(k+1)/2-1}N + \gamma_k\}$ where γ_n has yet to be determined. It is easily seen that in order to have the sets C_i disjoint, it is necessary and sufficient that for all m

$$\gamma_m \not\equiv \gamma_k \pmod{2^{k(k+1)/2}}$$

for $k = 2, 3, \cdots m-1$. Likewise, in order to have $C_m \subset D_k$ we must have

$$\gamma_m \equiv \gamma_k \pmod{2^{k(k+1)/2-1}}$$

for all m and for $k = 2, 3, \cdots m$. Combining these, we must solve the set of congruences

$$\gamma_m \equiv \gamma_k + 2^{k(k+1)/2-1} \pmod{2^{k(k+1)/2}}.$$

For initial conditions, we take $\gamma_1 = 0$, $\gamma_2 = 1$. A solution is then given by

$$\gamma_n = 1 + 2^2 + 2^5 + \cdots + 2^{n(n-1)/2-1}.$$

Our sets A_m and B_k become

$$A_m = \{2N\} \cup \{8N+1\} \cup \{64N+5\} \cup \{1024N+37\} \cup \cdots \cup C_m;$$
$$B_1 = \{N\} = I;$$
$$B_2 = \{2N\} \cup \{4N+1\};$$
$$B_3 = \{2N\} \cup \{8N+1\} \cup \{32N+5\}$$

and so on. Since the sets C_i are pairwise disjoint, $\Delta(A_m) = \sum_1^m \Delta(C_i) = \sum_1^m 2^{-i(i+1)/2}$, and as m increases this sum approaches the irrational number β whose dyadic expansion is $.10100100010\cdots$. Computing $\Delta(B_k)$ we have $\Delta(B_k) \leq \sum_1^{k-1} \Delta(C_i) + \Delta(D_k) = \sum_1^{k-1} 2^{-i(i+1)/2} + 2^{1-k(k+1)/2}$ which again approaches β as k increases. Finally, we set $S = UA_m = UC_m$. We certainly have $A_m \subset S \subset B_k$ for all m and k.

This construction can be carried out for any irrational number. If β is expressed in the dyadic form $\sum_1^\infty 2^{-\lambda_n}$, we choose C_m as $\{2^{\lambda_m}N + \gamma_m\}$ where $\gamma_1 = 0$, and $\gamma_{n+1} = (2^{\lambda_1} + 2^{\lambda_2} + \cdots + 2^{\lambda_n})/2$. Then, the set $S = UC_m$ belongs to $\mathfrak{D}_\mu$ and has measure $\mu(S) = \beta$. Recalling that as A ranges over the countable set $\mathfrak{D}_0$, $\mu(A)$ takes on every rational value, we have proved the following result.

THEOREM 6. *The set of values of $\mu(S)$ for S in $\mathfrak{D}_\mu$ is exactly the closed unit interval.*

An obvious consequence of this is that the class $\mathfrak{D}_\mu$ has cardinal number c, the cardinal of the continuum; moreover, the class $\mathfrak{D}_\mu/\mathfrak{K}$ where $\mathfrak{K}$ is the class of null sets, also has cardinal c. For comparison we recall that while there are 2^c Lebesgue measurable sets, the class of measurable sets modulo the null sets has cardinal c.

5. Quasi-progressions. In this section, $[x]$ will always denote the greatest integer in x, while $((x))$ will be the fractional part of x: $x = [x] + ((x))$. Let $A = \{a_n\}$ where $a_n = [\alpha n + \beta]$. If α is an integer, then $a_n = \alpha n + [\beta]$ and A is an arithmetic progression. If α is rational, then A is a finite union of progressions. If we suppose that $\alpha = p/q$, and write $u = mq + r$, where $1 \leq r \leq q$, we find that

$$A = \{pm + b_1\} \cup \{pm + b_2\} \cup \cdots \cup \{pm + b_q\}$$

where $b_r = [\beta + rp/q]$. The case of α irrational remains; the set A will

then be called a quasi-progression. We assume that $\alpha > 1$. We will show that such a set does not belong to $\mathfrak{D}_\mu$ and that moreover $\mu(A) = \mu(A') = 1$ so that A is extremal—i. e., has outer measure one and inner measure zero.

LEMMA 1. *Let A and B be two non-negative integers, $A \geq 2$, and α and β two positive real numbers with $0 \leq \beta < 1$. Suppose that we can find a rational number p/q with $(p, q) = 1$ such that*

(i) $$0 \leq \alpha - p/q \leq q^{-3/2}$$

(ii) $$\sqrt{q} > A(1 + \sqrt{3})/(2 - 2\beta).$$

Then, there exists an integer r, $0 < r \leq A$ and integers n and m with $0 < n \leq Aq$, $0 \leq m < p$ such that

(iii) $$pn - Aqm = Bq + r$$

(iv) $$[\alpha n + \beta] = Am + B.$$

Proof. Part (iii) is immediate; choose r so that $Bq + r$ is divisible by $(p, qA) = (p, A) \leq A$, and then choose n and m as the least positive solutions of $px - Aqy = Bq + r$. Now, put

$$\begin{aligned}\delta &= \{(\alpha n + \beta) - (Am + B)\}/n \\ &= \alpha - p/q + (np - Amq - Bq)/nq + \beta/n.\end{aligned}$$

Using (i) and (iii), we have

$$\begin{aligned}0 \leq \delta &\leq q^{-3/2} + r/nq + \beta/n \\ &\leq \left\{ \frac{A\sqrt{q} + A}{q} + \beta \right\} \frac{1}{n}.\end{aligned}$$

If (ii) holds then

$$\sqrt{q} > \frac{A + \{A^2 + 4A(1 - \beta)\}^{\frac{1}{2}}}{2(1 - \beta)}$$

and

$$(1 - \beta)q - A\sqrt{q} - A > 0$$

so that

$$0 \leq \delta < \{(1 - \beta) + \beta\}(1/n) = 1/n.$$

Hence,

$$0 \leq \alpha n + \beta - (Am + B) < 1,$$

and (iv) holds.

LEMMA 2. *If α is irrational and if we can choose an infinite number of distinct rationals p/q for which* (i) *and* (ii) *hold, then there are an infinite number of distinct pairs n, m, for which* (iv) *holds.*

Suppose that as we select p/q, only a finite number of distinct pairs n, m, appear. There must be an infinite number of distinct rationals p/q which correspond to the same pair n, m. Since the denominators of these rationals will increase without bound, (i) implies that $\alpha = \lim p/q$. From (iii), however, we see that $p/q = (Am + B)/n + r/nq$; in this, m and n are fixed and r, although it depends on p and q, always lies between 0 and A. Taking limits, we obtain $\alpha = (Am + B)/n$, a rational number.

Let us now introduce the notation $k(\alpha, \lambda)$ for the greatest lower bound of the numbers $q^\lambda(\alpha - p/q)$ taken over all rational numbers p/q, with $(p, q) = 1$, and $p/q < \alpha$. If α is algebraic of degree n, then $k(\alpha, n) > 0$, and if α is rational, $k(\alpha, \lambda) = 0$ for all λ. [3; 157, 160.]

LEMMA 3. *If α is irrational, and $\lambda < 2$ then $k(\alpha, \lambda) = 0$.*

This is well known for λ integral; when $\lambda = 1$, it is equivalent to stating that $\liminf ((n\alpha)) = 0$. For $\lambda < 2$, the usual proof by means of the successive convergents of the continued fraction expansion of α still suffices. [3, Theorem 171.]

THEOREM 7. *If $\alpha > 1$ is irrational, and β is any positive real number, then the set $A = \{a_n\}$ where $a_n = [\alpha n + \beta]$ has an infinite number of terms in common with any arithmetic progression, and has outer measure one, inner measure zero.*

From the lemmas, we can find an infinite number of distinct pairs n, m, for which $[\alpha n + \beta] = Am + B$, given any two integers A and B. In Lemma 1, we required β to lie between 0 and 1. This restriction is inessential, for if $\beta = b + \beta_0$ where $0 \leq \beta < 1$, then $[\alpha n + \beta] = b + [\alpha n + \beta_0]$. The proof that $\mu(A) = 1$ is the same as in Theorem 5. To show that A has inner measure zero, we must show that A contains no infinite set of $\mathfrak{D}_0$. Suppose $\{an + b\} \subset A$; there is then a sequence of integers λ_i such that $an + b = [\alpha\lambda + \beta]$ for all large n. Suppose that $\alpha > 2$; the case where $1 < \alpha < 2$ may be treated in much the same way. Taking first differences, we have, again for all large n, $a = \alpha(\lambda_{n+1} - \lambda_n) + \epsilon_n - \epsilon_{n+1}$ where $\epsilon_n = ((\alpha\lambda_n + \beta))$. Set $\lambda_{n+1} - \lambda_n = \theta_n$, integral, and let $\underline{\theta}$ and $\bar{\theta}$ be the minimum and maximum values of θ_n for large values of n. Then, $\alpha\theta_n = a + \epsilon_n - \epsilon_{n+1}$ and since $0 \leq \epsilon_n < 1$, $0 \leq \alpha(\bar{\theta} - \underline{\theta}) < 2$. But, $\alpha > 2$ so that $\underline{\theta} = \bar{\theta}$ and $\theta_n \equiv \theta$ for all large n; λ_n is then $n\theta + \lambda_0$ and $\epsilon_{n+1} - \epsilon_n$ is a constant c, for all large n. This in turn gives us $\epsilon_m = mc + \epsilon_0$ and unless c is zero, these would be unbounded. We have therefore shown that

$\epsilon_n = ((\alpha\lambda_n + \beta)) = ((\theta\alpha n + \alpha\lambda_0 + \beta))$ is constant. This contradicts the familiar assertion that if γ is irrational, the points $((\gamma m + d))$ are everywhere dense on $(0, 1)$. [3, 364]. Since $\mu(A) + \mu(A') = 2 \neq 1$, A is clearly not measurable.

6. Limit density. We now consider a more inclusive class of sets of integers in place of $\mathcal{D}_\mu$. Let $\mathcal{D}$ be the class of all sets $A = \{a_n\}$ where $a_n = \alpha n + o(n)$, for some positive real number $\alpha \geq 1$. This class clearly includes $\mathcal{D}_0$ and, as we shall prove shortly, it strictly includes $\mathcal{D}_\mu$ as well. It is the class with which the term density is usually associated. For an arbitrary set $A \subset I$, we first define two important functions. $\alpha(n)$ will denote the characteristic function of the set A, taking the value 1 for $n \in A$ and 0 for $n \notin A$. $A(n)$, the distribution function of the set A, is defined as $\sum_1^n \alpha(k)$ and is the number of integers in A not greater than n. $\beta(n)$ and $B(n)$ will be the corresponding functions for a set B. The characteristic function of the set $A \cap B$ is $\alpha(n)\beta(n)$, and that of $A \cup B$ is $\alpha(n) + \beta(n) - \alpha(n)\beta(n)$.

The quotient $A(n)/n$ is the density of the set A in the interval $(1, n)$; the number $\lim A(n)/n$ when it exists is denoted by $D(A)$ and is called the density of A. Clearly, the set $\{a_n\}$ where $a_n = \alpha n + o(n)$ has density $D(A) = 1/\alpha$. More generally, the following equivalence is easily shown: *if* $A = \{a_n\}$ *and* $A \in \mathcal{D}$, *then* $D(A) = \lim A(n)/n = \lim n/a_n = \lim (1/n) \sum_1^n \alpha(k)$.

We observe that the last expression is simply the Cesàro limit of the characteristic function $\alpha(n)$. We can therefore produce sets for which density is not defined by selecting any sequence of zeros and ones whose $(C, 1)$ limit fails to exist, and forming the set whose characteristic function it is.

Unfortunately, the class $\mathcal{D}$ is not a measure class and D is not a measure. Some of the measure properties hold; if A belongs to $\mathcal{D}$ so does A', and if A and B belong to $\mathcal{D}$ and are disjoint, then $A \cup B$ belongs to $\mathcal{D}$ and $D(A \cup B) = D(A) + D(B)$. However, if A and B are not disjoint, neither $A \cup B$ nor $A \cap B$ need belong to $\mathcal{D}$; [11, 16]. Consider, for example, the sets defined as follows: let $\Lambda = \{\lambda_n\}$ be a set of integers not having a density, let $A = \{3N\}$ and $B = \{b_n\}$ where $b_n = a_n$ if $n \in \Lambda$, and $b_n = 1 + a_n$ otherwise. Both A and B have density $1/3$, but $A \cap B = \{3\lambda_n\}$ which fails to have a density.

In spite of this, we can still define an outer density, $\omega(S)$ as $\inf D(A)$ taken over all sets A in $\mathcal{D}$ which contains S. Pólya has proved that this

outer density can be analytically expressed, and is in fact the Pólya maximum density; that is,

$$\omega(S) = \bar{D}_1(S) = \lim_{\theta \to 1} \limsup_{n \to \infty} \frac{S(n) - S(\theta n)}{n - \theta n}.$$

More precisely, Pólya proved that if $\bar{D}_1(S) = d$, then there exists a set B of $\mathfrak{D}$ having density d and containing S. [7, 562.] The minimum density $\underline{D}_1(S)$ is obtained if lim sup is replaced by lim inf. Then, $\underline{D}_1(S) + \bar{D}_1(S') = 1$, and if $\underline{D}_1(S) = d$, there is a set A in $\mathfrak{D}$ having density d and contained in S. Thus, Pólya minimum density is inner density, and is $1 - \omega(S')$.

If we attempt to extend $\mathfrak{D}$ in the usual manner by means of this outer density ω, we find that the extension is not proper. Let $\mathfrak{D}_\omega$ be all sets S for which $\omega(S) + \omega(S') = 1$.

THEOREM 8. $\mathfrak{D}_\omega = \mathfrak{D}$.

If S belongs to $\mathfrak{D}_\omega$, there must exist sets A and B in $\mathfrak{D}$ with $A \subset S \subset B$ and $D(A) = D(B) = \omega(S)$. The continued inclusion implies that $A(n) \leq S(n) \leq B(n)$, and dividing by n and taking limits, we find that lim $S(n)/n = \omega(S)$, and $S \in \mathfrak{D}$.

Since $\mathfrak{D}_0 \subset \mathfrak{D}$, it is true that $\mathfrak{D}_\mu \subset \mathfrak{D}$. In the previous section, we showed that the quasi-progression $\{[\alpha n + \beta]\}$ did not have a μ measure; since it clearly has density $1/\alpha$ we see that $\mathfrak{D}_\mu \subset \mathfrak{D}$ with strict inclusion. Moreover, $\mu(S) \geq \bar{D}_1(S)$ for all sets S.

Rather more useful than the maximum or minimum densities are the upper and lower densities, defined by:

$$UD(A) = \limsup A(n)/n$$

$$LD(A) = \liminf A(n)/n.$$

It is clear from the definitions that $D_1(A) \leq LD(A) \leq UD(A) \leq \bar{D}_1(A)$. All inequality signs may be strict as is shown by the example

$$\alpha(n) = (1, 1, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, \cdots)$$

for which $\bar{D}_1(A) = 1$, $UD(A) = 2/3$, $LD(A) = 1/3$, $\underline{D}_1(A) = 0$. If $C = A \cup B$, then $C(n) \leq A(n) + B(n)$ and $C(n) - C(\theta n) = \sum_{n\theta}^{n} \gamma(k) \leq \sum_{n\theta}^{n} \{\alpha(k) + \beta(k)\} = \{A(n) - A(\theta n)\} + \{B(n) - B(\theta n)\}$. Consequently, $UD(A \cup B) \leq UD(A) + UD(B)$ and $\bar{D}_1(A \cup B) \leq \bar{D}_1(A) + \bar{D}_1(B)$.

Suppose that we have a sequence of sets $A_1 \supset A_2 \supset A_3 \supset \cdots$ with $D(A_n) = d$ for all n. As observed in Section **4**, we cannot expect properties

of the sets to carry over to their intersection. For example, if $A_n = \{n, n+1, \cdots\}$, then $\bigcap A_n$ is void, while $D(A_n) = 1$. However, by suitably modifying the notion of the intersection of a family of sets, we can prove a rather strong theorem of this type. It is the same theorem that in Section **4** we proved did *not* hold for $\mathfrak{D}_\mu$. (Theorem 5.)

THEOREM 9. *If $A_1 \supset A_2 \supset A_3 \cdots$ then there exists a set B such that $B \dot{\subset} A_n$ for $n = 1, 2, \cdots$ and such that $LD(B) = \lim LD(A_n)$, $UD(B) = \lim UD(A_n)$. In particular, if $A_n \in \mathfrak{D}$, then $B \in \mathfrak{D}$ and $D(B) = \lim D(A_n)$.*

Proof. Since $\liminf A_k(n)/n = LD(A_k)$, and $\limsup A_k(n)/n = UD(A_k)$, we can choose a sequence of integers $n_1 < n_2 < n_3 \cdots$ such that $A_k(n)/n \geq (1 - (1/k+1))LD(A_k)$ for all $n > n_k$ and $A_k(n)/n \geq (1 - (1/k+1))UD(A_k)$ for $n = 1 + n_k$. That is, n_k is an integer such that the partial density of A_k is not too small beyond it, while it is large enough at least once. Let I_k be the segment composed of all n with $n_k < n \leq n_{k+1}$. We construct the set B by assigning to it all the integers of A_k lying in I_k; thus, $B \cap I_k = A_k \cap I_k$ for all k. We must show that B has all the desired properties. First, since $A_k \subset A_m$ for all $k > m$, and $I_k \neq 0$,

$$B = \bigcup B \cap I_k = \bigcup (A_k \cap I_k) \dot{\subset} A_m \cap \bigcup_{k \geq m} I_k \dot{\subset} A_m$$

and therefore $B \dot{\subset} A_m$, holding for any m. This implies that $UD(B) \leq \text{g.l.b. } UD(A_m) = \lim UD(A_m)$ and $LD(B) \leq \lim UD(A_m)$. We compute $B(n)/n$ and show that we can reverse these inequalities, thus establishing equality. If $n \in I_k$, then

$$B(n) = A_1(n_2) - A_1(n_1) + \cdots + A_{k-1}(n_k) - A_{k-1}(n_{k-1}) + A_k(n) - A_k(n_k).$$

Since $A_{r+1} \subset A_r$, $A_r(m) - A_{r+1}(m) \geq 0$ and

$$B(n) \geq A_k(n) - A_1(n_1).$$

For all n in I_k,

$$B(n)/n \geq A_k(n)/n - o(1) \geq (1 - (1/k+1))LD(A_k) - o(1)$$

and as n increases, $\liminf B(n)/n = LD(B) \geq \lim LD(A_k)$. For the upper density, we observe that there is an n in I_k for which $B(n)/n \geqq A_k(n)/n - o(1) \geqq (1 - (1/k+1))\, UD\,(A_k) - o(1)$ so that $\limsup B(n)/n = UD(B) \geq \lim UD(A_k)$.

COROLLARY. *If $A_1 \subset A_2 \subset A_3 \subset \cdots$ then there exists a set B such that $A_n \dot{\subset} B$ for all n, while $UD(B) = \lim UD(A_n)$ and $LD(B) = \lim LD(A_n)$.*

The sets B are not unique; however, if two sets B_1 and B_2 satisfy the conditions of the theorem, it is evident that the sets $B_1 - B_2$ and $B_2 - B_1$ will both be of density zero. Thus, if we identify two sets whose symmetric difference is null, these modified intersections of the sets A_n are unique. The identification yields a homomorphic image of the boolean ring of subsets of I whose kernel is the ideal of all sets having zero density. In view of this, it is perhaps important to point out that the set B can be chosen so that $A_n + B$, the symmetric difference, is finite and not merely null.

We now introduce the interval sequence associated with a set A. Suppose that A is formed in the following way: a block composed of the first g_1 integers, followed by a gap of length g_2, and then by a block of g_3 consecutive integers, and then a gap of length g_4, and so on. The sequence of integers $\{g_i\}$ determines the set A uniquely. g_1 may be zero, but all others are positive. Put $G_n = \sum_1^n g_i$.

THEOREM 10. $UD(A) = \limsup V_n/G_{2n-1}$

$LD(A) = \limsup V_n/G_{2n}$

where $V_n = g_1 + g_3 + \cdots + g_{2n-1}$.

The greatest values of $A(m)/m$ occur when m is the last term of a block of consecutive integers, and the least values when m is the last term of a void interval. In the first instance, $m = G_{2n-1}$ and $A(m) = g_1 + g_3 + \cdots + g_{2n-1} = V_n$, giving us the first relation above. For the second, $m = G_{2n}$ and $A(m)$ is again V_n.

THEOREM 11. $UD(A) \geq UD(A) \liminf G_{2n-1}/G_{2n} \geq LD(A)$

$\{1 - LD(A)\} \geq \{1 - LD(A)\} \liminf G_{2n}/G_{2n+1} \geq \{1 - UD(A)\}$.

The left hand sides of these inqualities are obvious since G_n is an increasing sequence. For the right half of the first,

$$UD(A) \liminf G_{2n-1}/G_{2n} = \limsup (V_n/G_{2n-1}) \liminf (G_{2n-1}/G_{2n}) \geq \liminf V_n/G_{2n} = LD(A).$$

In the second,

$$\{1 - LD(A)\} \liminf G_{2n}/G_{2n+1} \geq \liminf (1 - V_n/G_{2n})(G_{2n}/G_{2n+1}) \geq \liminf (G_{2n} - V_n)/G_{2n+} = \liminf (G_{2n+1} - V_{n+1})/G_{2n+1} \geq 1 - UD(A).$$

COROLLARY 1. *If* $A \in \mathfrak{D}$ *and* $0 < D(A) < 1$, *then* $\lim G_n/G_{n+1} = 1$.

For, $LD(A) = UD(A) = d$, and $d \neq 0$, $1 - d \neq 0$. Hence, by the theorem, $\liminf G_{2n-1}/G_{2n} = \liminf G_{2n}/G_{2n+1} = 1$; since $G_n/G_{n+1} \leq 1$, the result follows. It is clear that this limit need not exist if $D(A) = 0$ or 1.

COROLLARY 2. *If* $\liminf G_n/G_{n+1} = 0$, *then* $UD(A) = 1$, $LD(A) = 0$.

Far weaker assumptions are sufficient to yield extremal maximum and minimum densities.

THEOREM 12. *If* $\liminf G_n/G_{n+1} < 1$, *then* $\bar{D}_1(A) = 1$, $\underline{D}_1(A) = 0$.

From the hypothesis, $\limsup g_{2n-1}/G_{2n-1} > 0$ and $\limsup g_{2n}/G_{2n} > 0$. Then, for values of θ sufficiently close to 1, g_{2n-1}/G_{2n-1} and g_{2n}/G_{2n} exceed $1 - \theta$ for an infinite number of integers n. Put $N = G_{2n-1}$; since g_{2n-1} is the length of a filled block of integers, and is greater than $N(1 - \theta)$, $A(N) - A(N\theta) = N - N\theta$, so that the quotient of these has the value 1 for infinitely many N. Hence, $\bar{D}_1(A) = 1$. In the same manner, putting $N = G_{2n}$, and recalling that g_{2n} is the length of a gap, we find that $A(N) - A(N\theta) = 0$ infinitely often, and $\underline{D}_1(A) = 0$.

Sets having outer measure 1 and inner measure 0 are called extremal sets. For a limit density, A is extremal if $UD(A) = 1$, and $LD(A) = 0$. The possibility of expressing a space as a sum of disjoint extremal sets has been discussed for ordinary Lebesgue measure [10]. In this connection, the following theorem is of interest.

THEOREM 13. *The set* I *may be split into a countable collection of disjoint sets which are extremal in the sense of density. That is,* $I = \bigcup_i^{\infty} A_k$ *where* $A_n \cap A_m = 0$ *if* $n \neq m$, *and* $UD(A_k) = 1$, $LD(A_k) = 0$.

A set that is extremal in the sense of density is also extremal with respect to inner and outer density, and thus certainly extremal for the μ measure discussed in the previous sections of this paper. We construct the sets A_k so as to have very long blocks of consecutive integers, followed by very long gaps. Choose an interval sequence $\{g_i\}$ so that $\lim G_n/G_{n+1} = 0$. For example, take $g_n = n(n!)$. By Corollary 2, Theorem 11, the set corresponding to this will be extremal. We must next split up I around this set. To simplify the notation, we shall write $g(n)$ and $G(n)$ for g_n and G_n. Let $E(n)$ denote the interval of integers k satisfying $G(n-1) < k \leq G(n)$. Choose any ω^2

type ordering of the positive integers, $\lambda_{k,j}$, such that $1 + \lambda_{k,j} < \lambda_{k,j+1}$. For example, we could use the following:

$$(\lambda_{k,j}) = \begin{bmatrix} 1 & 3 & 6 & 10 & . & . & . \\ 2 & 5 & 9 & 14 & . & . & . \\ 4 & 8 & 13 & 19 & . & . & . \\ 7 & 12 & . & . & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

Then, the sets A_k are given by

$$A_k = \bigcup_{j=1}^{\infty} E(\lambda_{k,j}).$$

Since the intervals $E(n)$ are all disjoint, and the $\lambda_{k,j}$ all distinct, these sets are all disjoint; since the array $(\lambda_{k,j})$ contains all the integers, $\bigcup_1^{\infty} A_k = I$. The terms of A_k fall into blocks of the form $[G(\lambda_{k,j} - 1),\ G(\lambda_{kj})]$ and the greatest values of $A_k(n)/n$ must occur when n is a $G(\lambda_{kj})$. Computing this, we have

$$A_k(n)/n = \frac{g(\lambda_{k1}) + g(\lambda_{k2}) + \cdots + g(\lambda_{kj})}{G(\lambda_{kj})}$$

$$\geqq g(\lambda_{kj})/G(\lambda_{kj})$$

by virtue of the restriction on the ordering λ_{kj}. Then, since $\lim G_n/G_{n+1} = 0$, $\limsup A_k(n)/n \geq 1$ and $UD(A_k) = 1$. The union of two sets having upper density 1 also has upper density 1; since $A_k' = \bigcup_{n \neq k} A_n$, $UD(A_k') = 1$ and therefore $LD(A_k) = 0$.

7. Generalized limit densities. Perhaps the most apparent method of generalizing the density discussed in the previous section is to replace the operation 'lim' in the definition $D(A) = \lim A(n)/n$ by a generalized limit. We can always get a total measure density—a measure defined for *all* subsets of the space—by writing $D_L(A) = \text{LIM}\, A(n)/n$ where LIM is a limit defined for all bounded sequences [1, 34]. Otherwise expressed, let L be any linear functional on the space of all bounded sequences such that $L(1) = 1$, where $1 = (1, 1, 1, 1, \cdots)$ and define $\Delta_L(A)$ to be $L(\alpha)$ where $\alpha = (\alpha(n))$ is the characteristic function of the set A [4]. L can here be chosen as an extension of the functional Δ defined over the linear span of the characteristic functions of the sets in $\mathfrak{D}_0$ and the resultant measure density will be an extension of Δ to all subsets of I [1, 231].

More useful, perhaps, are the limit densities obtained by replacing

ordinary limit by a regular summability method; let T be a transformation of the space of sequences given by a real or complex matrix (a_{nv}) so that if $x = (x_i)$, and $y = (y_i)$ and $y = T(x)$, then $y_n = \sum_{v=1}^{\infty} a_{nv}x_v$. We define a generalized limit by writing $T\text{-}\lim x = \lim y$, whenever this ordinary limit exists. T is regular if this limit is an extension of the usual limit; the Toeplitz necessary and sufficient conditions for regularity are [9]

(i) $\sum_{v=1}^{\infty} | a_{nv} | \leq M$, for all n

(ii) $\lim_{n\to\infty} a_{nv} = 0$, for all v

(iii) $T\text{-}\lim (1) = 1$.

Instead of taking $D(A)$ to be $\lim A(n)/n$ we express this in terms of the characteristic function and write this as $(C, 1)\text{-}\lim \alpha$. In order to actually have an extension of ordinary density, we shall require that T be regular over $(C, 1)$. This is equivalent to requiring that

(iv) $\sum_{v=1}^{\infty} v \, | a_{nv} - a_{n,v+1} | \leq K$, all n.

We now define our generalized density by $D_T(A) = T\text{-}\lim \alpha$, and denote the class of sets for which this exists by $\mathfrak{D}_T$. Clearly, $\mathfrak{D} \subset \mathfrak{D}_T$. We define upper and lower T density in a corresponding fashion as $UD_T(A) = T\text{-}\lim \sup \alpha$, and $LD_T(A) = T\text{-}\lim \inf \alpha$, where $T\text{-}\lim \sup x$ is $\lim \sup T(x)$. Theorems on regular transformations will now give rise to corresponding theorems on generalized densities. We first prove that no generalized limit density can be total.

THEOREM 14. *The class $\mathfrak{D}_T$ does not contain all the subsets of I.*

We must produce a set of integers not in $\mathfrak{D}_T$. This is equivalent to finding a characteristic function $\alpha(n)$ whose T limit does not exist. A classical theorem of Steinhaus asserts that given any regular transformation T, there is a sequence of zeros and ones not summable T [8].

Let B denote all bounded real sequences, C all convergent sequences, N all null sequences, and Z all sequences of zeros and ones. If x and y are two sequences xy will be the sequence $(x_n y_n)$; we write $x \leqq y$ if $x_n \leqq y_n$ for all n.

THEOREM 15. *If $x \in B$, $T(x) \in C - N$ then there is a sequence z in Z such that $T(xz)$ is not in C.*

Consider the transformation T_x defined by $T_x(y) = T(xy)$. Its matrix is (b_{nv}) where $b_{nv} = x_v a_{nv}$. Since $x \in B$, conditions (i) and (ii) for regularity are satisfied. Since $T(x) \in C - N$, $\lim T(x) = T_x\text{-lim } (1) = c$ exists, and is not zero; the transformation $(1/c)T_x$ is then regular, and there is then a sequence of zeros and ones z for which $T_x(z) = T(xz)$ does not converge.

COROLLARY. *Given T, and a set A in $\mathfrak{D}_T$ with $D_T(A) > 0$, there is a set B contained in A and not in $\mathfrak{D}_T$.*

For, if α is the characteristic function of A, then αz is the characteristic function of a set B contained in A.

We introduce the symbol V for the special transformation which takes the sequence $(x_1, x_2, \cdots)$ into the sequence $(x_2, x_3, \cdots)$.

LEMMA. *Let $\beta \in Z$, $\alpha \in Z$, with $\alpha \leq \beta$. Then, if $\beta V(\beta) = 0$ there exists a sequence β^* in Z such that $\alpha = \beta\beta^*$ while $(C, 1)$-lim $(\beta^* - \beta) = 0$.*

Choose β^* as $\alpha + V(\beta) - V(\alpha)$. Since $\alpha \leq \beta$, $\beta\alpha = \alpha$ and $\beta V(\alpha) = \beta V(\beta) V(\alpha) = 0$. Hence, $\beta\beta^* = \beta\alpha + \beta V(\beta) - \beta V(\alpha) = \alpha$. Since $\beta^*(n) = \alpha(n) + \beta(n+1) - \alpha(n+1)$, investigating the possible values of each term, $\beta^*(n)$ is seen to be always either zero or one, and $\beta^* \in Z$. If $x \in B$, then $V(x) - x$ is always $(C, 1)$ summable to zero; since $\beta^* - \beta = \alpha - \beta + V(\beta) - V(\alpha) = \{V(\beta) - \beta\} - \{V(\alpha) - \alpha\}$, we conclude that $(C, 1)$-lim $(\beta^* - \beta) = 0$.

We use this to prove that a regular summability method cannot preserve products, if it is stronger than $(C, 1)$.

THEOREM 16. *If $x \in Z$, $xV(x) = 0$, and $T(x) \in C - N$, then there is a sequence y in Z with $T(y) \in C$ but $T(xy) \notin C$.*

By Theorem 15, there is a sequence z in Z such that $T(xz) \notin C$. Applying the lemma, with $\beta = x$, and $\alpha = xz$ we set $y = \beta^*$. Since $\beta\beta^* = xy = \alpha = xz$, $T(xy) \notin C$. Since T is regular over $(C, 1)$, T-lim $(y - x) = 0$, so that $T(y) \in C$.

COROLLARY. *Let $A \in \mathfrak{D}_T$ with $D_T(A) > 0$; suppose that A does not contain a pair of consecutive integers. Then there is a set B also in $\mathfrak{D}_T$ but such that $A \cap B$ and $A \cup B$ do not belong to $\mathfrak{D}_T$.*

This is a generalization of the counterexample given for ordinary density in the previous section. In fact, we may again take A as $\{3N\}$. It is therefore true that no generalized density of this type can also be a measure density. Let us now assume that T is positive so that if x is positive, T-lim inf $x \geq 0$.

THEOREM 17. *The Carathéodory closure of $\mathfrak{D}_T$ is again itself.*

Suppose that S belongs to the closure of $\mathfrak{D}_T$. Then, for any positive ε, we can find sets A and B in $\mathfrak{D}_T$ such that $A \subset S \subset B$ and $D_T(B) - D_T(A) < \varepsilon$. Since $\alpha \leq \sigma \leq \beta$, $D_T(A) \leq T\text{-lim inf } \sigma \leq T\text{-lim sup } \sigma \leq D_T(B)$ and therefore $UD_T(S) - LD_T(S) < \epsilon$; letting ϵ approach zero, the upper and lower T densities of S must be equal, and S belongs to $\mathfrak{D}_T$.

If we relax the restriction (iv) that T be stronger than $(C, 1)$, we obtain a generalized density, now, however, not an extension of ordinary density. Corresponding questions may be discussed. Thus, in Theorem 16, we can replace the condition that T be stronger than $(C, 1)$ by the condition

$$\lim_{n\to\infty} \sum_{v=1}^{\infty} | a_{n,v+1} - a_{n,v} | = 0$$

since this is necessary and sufficient that T-lim $\{V(x) - x\} = 0$ for all x in B. It would also be of interest to know if Theorems 9 and 13 carry over to these generalized densities.

8. The dyadic mapping. Any total measure on the class of subsets of I defines a mapping of this class into the interval $[0, 1]$. Conversely, a measure is such a mapping, which in addition is additive for disjoint sets. Let us consider the function Γ given by $\Gamma(A) = \sum_{1}^{\infty} \alpha(n) 2^{-n}$, where $\alpha(n)$ is the characteristic function of A. This maps the set A onto the real number whose dyadic expansion is given by the sequence of zeros and ones α. It is evident that $\Gamma(A \cup B) + \Gamma(A \cap B) = \Gamma(A) + \Gamma(B)$ and that if $A \subset B$, $\Gamma(A) \leq \Gamma(B)$. Γ is a total measure defined on I; it differs from the ones described in the previous section in that finite sets do not have zero measure. This is also true of any measure given by a linear functional of the form $L(\alpha) = \Sigma \alpha(n) c_n$. Suppose that $\Gamma(A) = \Gamma(B)$. In terms of α and β this means that if $A \neq B$, there must be an n_0 such that $\alpha(n) = \beta(n)$ for $n < n_0$, $\alpha(n_0) = 1$, $\beta(n_0) = 0$, and $\alpha(n) = 0$, $\beta(n) = 1$ for $n > n_0$. Returning to the sets, this implies that $A \doteq 0$, $B \doteq I$. Thus, if we consider only infinite subsets of I, Γ is a 1:1 mapping of this class onto $(0, 1)$ and at most 2:1 for the class of all subsets.

We can define a measure on classes of sets of integers by the usual Lebesgue measure of the corresponding sets of points on the unit interval into which these classes are mapped by Γ. For example, consider the class $\mathfrak{D}_0$. Each set A of this is expressible as the union of a finite number of arithmetic progressions with the possible exception of a finite number of points. The

function $\alpha(n)$ is periodic for large n, and $\Gamma(A)$ is rational. Conversely, each rational point of $[0,1]$ is given by a set of $\mathfrak{D}_0$. Thus, $\Gamma(\mathfrak{D}_0)$ is precisely the set of rational points of $[0,1]$. (This affords us another simple proof of the fact that $\mathfrak{D}_0$ is countable.)

In a similar fashion we may investigate $\Gamma(\mathfrak{D}_\mu)$. Since $\mathfrak{D}_\mu$ contains infinite sets of measure zero, and any subset of such a set is also measurable, $\mathfrak{D}_\mu$ has cardinal c as we have observed before. In Section **5**, we proved that a quasi-progression with irrational difference was not measurable μ. If S is a subset of such a set A, then either S or $A - S$ must also be non-measurable, and there are c non-measurable sets. Thus, $\Gamma(\mathfrak{D}_\mu)$ is a non-countable set on the unit interval containing the rational points whose complement is also non-countable. It would be of interest to know if this set is measurable, and if so, whether its measure is zero or one.

The set $\Gamma(\mathfrak{D})$ is more easily discussed; since A belongs to $\mathfrak{D}$ only when the sequence α is $(C,1)$ summable, classical results show that $\Gamma(\mathfrak{D})$ has measure 1, and is of first category in the unit interval [2]. Since $\mathfrak{D}_\mu \subset \mathfrak{D}$, $\Gamma(\mathfrak{D}_\mu)$ too is of first category. Moreover, in this sense, almost every set of integers has density 1/2. Similarly, since $\mathfrak{D} \subset \mathfrak{D}_T$, $\Gamma(\mathfrak{D}_T)$ also has measure 1; however, this set too may be shown to be of first category [5].

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AN INEQUALITY FOR POTENTIAL FUNCTIONS.*

By K. O. Friedrichs.

In this paper we shall prove the following inequality: To every domain $\mathfrak{R}$, of a character to be specified, in the N-dimensional space of N variables $x_1, \cdots, x_N$, there is a constant $\Gamma > 0$ such that

$$(1) \qquad \int_{\mathfrak{R}} \phi_1^2 dx \leq \Gamma \int_{\mathfrak{R}} [\phi_2^2 + \cdots + \phi_N^2] dx; \qquad dx = dx_1 \cdots dx_N,$$

holds for the derivatives

$$\phi_k = \partial\phi/\partial x_k$$

of every potential function $\phi(x_1, \cdots, x_N)$ with finite Dirichlet integral,

$$\int_{\mathfrak{R}} \Sigma_\mu \phi_\mu^2 dx < \infty,$$

satisfying the condition

$$(2) \qquad \int_{\mathfrak{R}} \phi_1 dx = 0.$$

Inequality (1) also holds, with an appropriate constant $\Gamma_\bullet$, if condition (2) is replaced by

$$(2)_\bullet \qquad \int_{\mathfrak{R}_\bullet} \phi_1 dx = 0,$$

$\mathfrak{R}_\bullet$ being a fixed proper subdomain of $\mathfrak{R}$.

It is clear that the inequality would not hold if condition (2) were omitted, since the potential function $\phi = x_1$ would be a counter-example. If, on the other hand, terms were omitted from the right hand side of (1) the resulting inequality would also not hold: If, for example, ϕ_2^2 were omitted, the function $\phi = x_2^2 - (x_1 - c_1)^2$ would be counter-example, the constant c_1 being so chosen that condition (2) or $(2)_\bullet$ is satisfied.

The inequality (1) is a generalization of the inequality

$$\iint_{\mathfrak{R}} u^2 dxdy \leq \Gamma \iint_{\mathfrak{R}} v^2 dxdy$$

* Received March 21, 1946.

for the real and imaginary parts of an analytic function $u + iv$ of the complex variable $x + iy$ under the condition

$$\int\int_{\mathcal{R}} u dx dy = 0.$$

This inequality was proved earlier by the author [1]; it may be considered an analogue of the inequality

$$\int_{\mathcal{B}} u^2 ds \leq \Gamma \int_{\mathcal{B}} v^2 ds$$

if

$$\int_{\mathcal{B}} u ds = 0$$

for the boundary $\mathcal{B}$ of a circle, proved by Hilbert [2] and Lichtenstein [3], and generalized by M. Riesz [4], [5].

Our inequality is one of a class of inequalities which can be proved by similar methods. We mention *Korn's inequality* which we shall prove in a forthcoming paper.[2] It refers to systems $[u_1, \cdots, u_N]$ of N functions in $\mathcal{R}$ with finite Dirichlet integral and involves the symmetric and the antisymmetric part of the gradient tensor $u_{k/l} = \partial u_k / \partial x_l$:

$$s_{kl} = \tfrac{1}{2}(u_{k/l} + u_{l/k}), \qquad r_{kl} = \tfrac{1}{2}(u_{k/l} - u_{l/k}).$$

The statement is: To every domain $\mathcal{R}$, of a character to be specified, there is a constant $\mathrm{K} > 0$ such that

$$\int_{\mathcal{R}} \Sigma_{\kappa,\lambda} r_{\kappa\lambda}^2 dx \leq \mathrm{K} \int_{\mathcal{R}} \Sigma_{\kappa,\lambda} s_{\kappa\lambda}^2 dx$$

holds for all functions $[u_1, \cdots u_N]$ which satisfy the side condition

$$\int_{\mathcal{R}} r_{kl} dx = 0, \qquad k, l = 1, \cdots, N.$$

(The functions u_k are not required to satisfy a differential equation.) This inequality, given by Korn in 1909, furnishes an estimate of the Dirichlet integral by its "symmetric part":

$$\int_{\mathcal{R}} \Sigma_{\kappa,\lambda} u_{\kappa/\lambda}^2 dx \leq (\mathrm{K} + 1) \int_{\mathcal{R}} \Sigma_{\kappa,\lambda} s_{\kappa\lambda}^2 dx$$

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[1] See [1] in the Bibliography. The present proof, when specialized to $N = 2$, is simpler than that given in [1].

[2] On the Boundary Value Problems of the Theory of Elasticity and Korn's Inequality.

and that is the reason why it plays such a decisive part in the theory of elasticity.

Our inequality (1) under the condition (2) and also Korn's inequality will be established for *domains* $\mathfrak{R}$ characterized by a particular set of requirements: Firstly, $\mathfrak{R}$ is open and connected. Next, there exists a vector $\Omega_k(x_1, \cdots, x_n)$ with the following properties:

1. Ω_k is continuous in $\mathfrak{R} + \mathfrak{B}$, $\mathfrak{B}$ being the boundary of $\mathfrak{R}$,
2. $\Omega_k = 0$ on $\mathfrak{B}$,
3. Ω_k has continuous first derivatives $\Omega_{k/m} = \partial\Omega_k/\partial x_m$ in $\mathfrak{R}$,
4. These derivatives are bounded in $\mathfrak{R}$:

$$|\Omega_{k/m}| \leq C_1 \quad \text{in } \mathfrak{R}.$$

Denoting by $\mathfrak{R}_\delta$ the subdomain of all points in $\mathfrak{R}$ whose distance from $\mathfrak{B}$ is greater than δ we require:

5. For a certain $\nu > 0$, for which $\mathfrak{R}_\nu$ is connected, we have

$$\Omega_{\kappa/\kappa} \geq 1 \quad \text{in } \mathfrak{R} - \mathfrak{R}_\nu.$$

It is finally required that for every $\delta > 0$, $\delta \leq \nu$, there exist a function H

$$\mathrm{H} = \mathrm{H}^\delta(x_1, \cdots x_N)$$

with continuous first derivatives H_k such that

1. $\mathrm{H} = 1$ in $\mathfrak{R}_\delta$,
2. $\mathrm{H} \geq 0$ in $\mathfrak{R}$,
3. $\mathrm{H} = 0$ in $\mathfrak{R} - \mathfrak{R}_{\delta'}$ for an appropriate positive $\delta' < \delta$,
4. $|\mathrm{H}_k| \leq C_2/\delta$, with an appropriate constant C_2 (depending on $\mathfrak{R}$).

Domains satisfying these requirements will be called Ω-domains. In the second section of this paper we shall show that a domain whose boundary possesses a continuously differentiable normal-vector is an Ω-domain. We shall also show that domains with corners or edges, such as parallelopipeds, are Ω-domains.

We mention that these domains $\mathfrak{R}$ treated in **2** have an additional property, viz. that there exists a set of Ω-domains $\mathfrak{R}^{(\delta)}$ for $0 \leq \delta \leq \delta_0$ (with a certain $\delta_0 > 0$) with $\mathfrak{R}^{(0)} = \mathfrak{R}$, and $\mathfrak{R}_\delta \subset \mathfrak{R}^{(\delta)} \subset \mathfrak{R}$, such that

the derivatives of the corresponding functions $\Omega^{(\delta)}$ are uniformly bounded, (i. e., independent of δ in $0 \leq \delta \leq \delta_0$). Domains enjoying this property will be called special Ω-domains. The proof of inequality (1) presented in 1 will imply that also the constants $\Gamma^{(\delta)}$ in inequality (1) are uniformly bounded for such domains. Then we have as an immediate consequence the

THEOREM. *Let $\mathcal{R}$ be a special Ω-domain. Then for every potential function ϕ with continuous derivatives for which $\int_{\mathcal{R}} [\phi_2^2 + \cdots + \phi_N^2]dx$ is finite, also $\int_{\mathcal{R}} \phi_1^2 dx$ is finite.*

This theorem is of special interest for the case $N = 2$ where it implies that for every analytic function $w = u + iv$ of $z = x + iy$ for which $\int\int_{\mathcal{R}} u^2 dxdy$ is finite for a special Ω-domain $\mathcal{R}$, also $\int\int_{\mathcal{R}} v^2 dxdy$ is finite.[3]

1. Proof of the inequality (1). The proof of our inequality proceeds in four steps. The first decisive step consists in establishing the inequality

$$\int_{\mathcal{R}} \phi_1^2 dx \leq \Gamma_1 \int_{\mathcal{R}_v} \phi_1^2 dx + \Gamma_1 \int_{\mathcal{R}} \sum_{\nu=2}^{N} \phi_\nu^2 dx; \tag{1.01}$$

more precisely, we shall show that to every $\mathcal{R}_v$ there is a constant $\Gamma > 0$ such that (1.01) holds for all potential functions with finite Dirichlet integral. The later steps are concerned with establishing the inequality

$$\int_{\mathcal{R}_v} \phi_1^2 dx \leq \Gamma_2 \int_{\mathcal{R}} \sum_{\nu=2}^{N} \phi_\nu^2 dx \tag{1.02}$$

under condition (2).

The proof of the inequality (1.01) is derived from *the identity*

$$\begin{aligned} O^m &\equiv (\phi^2{}_m - \sum_{\kappa \neq m} \phi_\kappa{}^2)_m + 2 \sum_{\kappa \neq m} (\phi_\kappa \phi_m)_\kappa \\ &\equiv 2 \Sigma_\kappa (\phi_\kappa \phi_m)_\kappa - (\Sigma_\kappa \phi_\kappa{}^2)_m \\ &\equiv 2\phi_m \Sigma_\kappa \phi_{\kappa\kappa} = 0 \end{aligned} \tag{1.03}$$

valid for potential functions $\phi(x_1, \cdots x_N)$. Subscripts $m, \cdots$ here and

[3] This statement together with (1) entails: If a potential function u can be approximated in the mean by potential functions with continuous derivatives in $\mathcal{R} + \mathcal{B}$, then the same is true for the conjugate function. For this statement under conditions for the boundary somewhat different from ours in particular also for star-domains, see N. Aronszajn [6].

throughout indicate differentiation with respect to x_m; (except for the components Ω, for which the subscripts indicating differentiation are separated by a stroke).

With the abbreviation

$$\sum_{\mu}' = \sum_{\mu \neq 1}, \qquad \sum_{\mu,\nu}'' = \sum_{\mu \neq 1, \nu \neq 1} \tag{1.04}$$

we consider the relation

$$\int_{\mathcal{R}} \{\Omega_1 O^1 - \Sigma'_\mu \Omega_\mu O^\mu\} dx = 0, \tag{1.05}$$

which, if integration by parts could be carried out, would go over into

$$\int_{\mathcal{R}} \{\Omega_{1/1}(\phi_1^2 - \Sigma'_\nu \phi_\nu^2) + 2\Sigma'_\nu \Omega_{1/\nu}\phi_1\phi_\nu + \Sigma'_\mu \Omega_{\mu/\mu}(\phi_1^2 + \Sigma'_\nu \phi_\nu^2) - 2\Sigma'_\mu \Omega_{\mu/1}\phi_\mu\phi_1 - 2\Sigma''\Omega_{\mu/\nu}\phi_\mu\phi_\nu\} dx = 0,$$

or

$$\int_{\mathcal{R}} \Sigma_\kappa \Omega_{\kappa/\kappa}\phi_1^2 dx - \int_{\mathcal{R}} [2\Sigma'_\mu(\Omega_{\mu/1} - \Omega_{1/\mu})\phi_\mu\phi_1 + (\Omega_{1/1} - \Sigma'_\mu \Omega_{\mu/\mu})\Sigma'_\nu \phi_\nu^2 + 2\Sigma''_{\mu,\nu}\Omega_{\mu/\nu}\phi_\mu\phi_\nu] dx = 0. \tag{1.06}$$

That integration by parts is valid is not obvious, however, since no asumptions were made about the behavior of the derivatives ϕ_k at the boundary $\mathcal{B}$, except that the Dirichlet integral be finite. Instead of relation (1.05) we, therefore, consider first the relation

$$\int_{\mathcal{R}} H[\Omega_1 O^1 - \Sigma'_\mu \Omega_\mu O^\mu] dx = 0, \tag{1.05}'$$

in which $H = H^\delta$ is the function postulated in the introduction. Since $H^\delta = 0$ in $\mathcal{R} - \mathcal{R}_\delta$, we may integrate by parts and obtain a relation which differs from (1.06) only in that $\Omega_{k/m}$ is replaced by $(H^\delta \Omega_k)_m$. These derivatives are bounded in $\mathcal{R} - \mathcal{R}_\delta$. For $\Omega_{k/m}$ is assumed to be bounded; further, since $\Omega_k = 0$ on $\mathcal{B}$, $\delta^{-1}\Omega_k$ is bounded in $\mathcal{R} - \mathcal{R}_\delta$, hence also $H^\delta{}_m \Omega_k = (\delta H^\delta{}_m)(\delta^{-1}\Omega_k)$ by assumption 4 on H^δ. Since $H^\delta = 1$ in $\mathcal{R}_\delta$ we obtain for the left member of (1.06), extended over $\mathcal{R}_\delta$ instead of $\mathcal{R}$, the estimate

$$\Gamma_3 \int_{\mathcal{R} - \mathcal{R}_\delta} \Sigma_\mu \phi_\mu^2 dx,$$

which approaches zero with δ. Therefore relation (1.06) is valid.

From relation (1.06) and properties 4 and 5 of Ω_k we obtain, after using Schwarz's inequality,

$$\int_{\mathfrak{R}-\mathfrak{R}_\nu} \phi_1^2 dx \leq \Gamma_4 \int_{\mathfrak{R}_\nu} \phi_1^2 dx + \Gamma_4 \int_{\mathfrak{R}} \Sigma'_\nu \phi_\nu^2 dx + \Gamma_3 \{\int_{\mathfrak{R}} \phi_1^2 dx \int_{\mathfrak{R}} \Sigma'_\nu \phi_\nu^2 dx\}^{\frac{1}{2}}$$

whence, after a few obvious steps,

$$(1.07) \qquad \int_{\mathfrak{R}} \phi_1^2 dx \leq 2(\Gamma_4 + 1) \int_{\mathfrak{R}_\nu} \phi_1^2 dx + (2\Gamma_4 + \Gamma_3^2) \int_{\mathfrak{R}} \Sigma'_\nu \phi_\nu^2 dx.$$

which is equivalent to (1.10).

To prove our theorem we therefore need only estimate $\int_{\mathfrak{R}_\nu} \phi_1^2 dx$ in terms of $\int_{\mathfrak{R}} \Sigma'_\nu \phi_\nu^2 dx$. This could be done in several ways. Rather directly we proceed in three steps as follows:

First we employ *Poincaré's inequality* in a restricted sense:[4] There is a constant $C_\bullet$ such that

$$\int_{\mathfrak{R}_\nu} \psi^2 dx \leq C_\bullet \int_{\mathfrak{R}_\nu} \Sigma_\mu \psi_\mu^2 dx$$

holds for all functions ψ possessing a finite Dirichlet integal over $\mathfrak{R}$ and satisfying the relation

$$\int_{\mathfrak{R}_\bullet} \psi dx = 0,$$

where $\mathfrak{R}_\bullet$ is a subdomain of $\mathfrak{R}_\nu$. We apply this inequality to $\psi = \phi_1$ and obtain

$$(1.08) \qquad \int_{\mathfrak{R}_\nu} \phi_1^2 dx \leq C_\bullet \int_{\mathfrak{R}_\nu} \Sigma_\mu \psi_{1\mu}^2 dx$$

$$(1.09) \qquad \int_{\mathfrak{R}_\bullet} \phi_1 dx = 0.$$

Poincaré's inequality estimates the integral of the square of the function in terms of the integral of the sum of the squares of the derivatives. For potential functions estimates in the opposite direction are possible: the integral of the sum of the squares of the derivatives can be estimated through the integral of the square of the function over a wider domain. More specifically, we need an estimate of the integral

$$\int_{\mathfrak{R}_\nu} \Sigma_\mu \phi_{1\mu}^2 dx \quad \text{through} \quad \int_{\mathfrak{R}_\nu} \Sigma'_\nu \phi_\nu^2 dx.$$

[4] See [6] Courant-Hilbert, vol. II, ch. VII, §§ 3.1, 6, 3.8. At this place the assumption is used that $\mathfrak{R}_\nu$ is connected.

To this end we employ the identity

$$(1.10)\qquad \begin{aligned}(\Sigma'_\nu\phi_\nu^2)_{11} - \Sigma'_\mu(\Sigma'_\nu\phi_\nu^2)_{\mu\mu} + 2\,\Sigma''_{\mu,\nu}(\phi_\mu\phi_\nu)_{\mu\nu}\\ = 2[\Sigma'_\nu\phi_{\nu_1}{}^2 - \Sigma''_{\mu,\nu}\phi_{\mu\mu}\phi_{\nu\nu}] + 2[\Sigma'_\nu\phi_\nu\phi_{\nu 11} + \Sigma''_{\mu,\nu}\phi_\nu\phi_{\mu\mu\nu}]\\ = 2\,\Sigma_\kappa\phi_{\kappa 1}{}^2,\end{aligned}$$

which is valid for potential functions ϕ. Let $H = H^\nu$ be the function postulated in the introduction which vanishes in $\mathcal{R} - \mathcal{R}_\nu$. Then we multiply (1.10) by H, integrate over $\mathcal{R}$, and apply integration by parts. Thus we obtain

$$(1.11)\qquad \begin{aligned}2\int_{\mathcal{R}} H\Sigma_\kappa\phi_{\kappa 1}{}^2dx = \int_{\mathcal{R}} [(H_{11} - \Sigma'_\mu H_{\mu\mu})\Sigma'_\nu\phi_\nu^2\\ + 2\,\Sigma''_{\mu,\nu}H_{\mu\nu}\phi_\mu\phi_\nu]dx.\end{aligned}$$

Since $H = 1$ in $\mathcal{R}_\nu$, $H \geqq 0$ in $\mathcal{R}$, and the second derivative H_{mn} are bounded we can derive from (1.11) the relation

$$(1.12)\qquad \int_{\mathcal{R}_\nu} \Sigma_\kappa\phi_{\kappa 1}{}^2dx \leq \Gamma_5 \int_{\mathcal{R}_\nu} \Sigma'_\nu\phi_\nu^2dx.$$

Combining (1.12) with (1.08) and (1.01) we find

$$(1.13)\qquad \int_{\mathcal{R}} \phi_1{}^2dx \leq \Gamma_6 \int_{\mathcal{R}} \Sigma'\phi_\nu^2dx,$$

if (1.09) or (2)$_*$ for a proper subdomain $\mathcal{R}_*$ holds. If the side condition (2) holds for the whole domain $\mathcal{R}$ we assign to the function ϕ a constant c such that $\phi^* = \phi + cx$ satisfies (1.09). Inequality (1.13) then holds for ϕ^*. Since by (2)

$$\int_{\mathcal{R}} \phi_1{}^2dx = \int_{\mathcal{R}} (\phi^*_1)^2dx - c^2\int_{\mathcal{R}} x^2dx \leq \int_{\mathcal{R}} (\phi^*_1)^2dx$$

and the right hand member in (1.13) is the same for ϕ as for ϕ^*, we see that (1.13) also holds for the function ϕ satisfying relation (2). Thus our inequality (1) is proved.

2. On the Admitted Domains. In this section we shall discuss simple conditions for domains $\mathcal{R}$ to be what we have called Ω-domains in the introduction.

We first state: The domain $\mathfrak{R}$ is an Ω-domain if there exists an inside neighborhood $\mathfrak{S}^0$ of its boundary $\mathfrak{B}$ which is a one-to-one image of a spherical shell $\rho_0 \leq \rho \leq 1$, $\rho^2 = \Sigma_\kappa \xi_\kappa^2$, of a $[\xi_1, \cdots \xi_N]$-space given by functions $x = x(\xi)$ which possess continuous first derivatives $x_{l/m} = \partial x_l / \partial \xi_m$ in $\rho_0 \leq \rho \leq 1$ such that the jacobian does not vanish:

$$\det x_{l/m} \neq 0 \quad \text{in} \quad \rho_0 \leqq \rho \leqq 1. \tag{2.01}$$

The boundary $\mathfrak{B}$ is supposed to be the image of $\rho = 1$.

To prove this statement we have to construct functions Ω_k and H^δ, satisfying the specifications set up in the introduction.

We first observe that the mapping has an inverse $\xi_k = \xi_k(x)$ with continuous derivatives $\xi_{k/l} = \partial \xi_k / \partial x_l$ in $\mathfrak{S}^0$. We have

$$\Sigma_\lambda \xi_{k/\lambda} x_{\lambda/m} = \delta_{km}. \tag{2.02}$$

If the function $x_l(\xi)$ possessed continuous second derivatives $x_{l/mn}$ we could simply set

$$\Omega_l = \Sigma_\mu \xi_\mu (1 - \rho^{-N}) x_{l/\mu},$$

because then

$$\Sigma_\lambda \Omega_{\lambda/\lambda} = N + (1 - \rho^{-N}) \Sigma_{\lambda,\mu,\nu} \xi_\mu \xi_{\nu/\lambda} x_{\lambda/\mu\nu}$$

would be greater than 1 in the neighborhood of $\rho = 1$, ($N \geq 2$ being assumed).

Without assuming the existence of derivatives $x_{l/mn}$ we proceed as follows. Clearly, we can approximate the functions $x_l(\xi)$ by functions $X_l(\xi)$ defined in a shell $\rho_1 \leq \rho \leq 1$, $\rho_0 < \rho_1 < 1$, which possesses continuous second derivatives $X_{l/mn}$ and whose first derivatives $X_{l/mn}$ approximate $x_{l/m}$ as closely as desired. For our purpose it is sufficient to require

$$\begin{aligned} &\left| \Sigma_{\kappa,\lambda,\mu} \xi_\mu \xi_\kappa \xi_{\kappa/\lambda} (X_{\lambda/\mu} - x_{\lambda/\mu}) \right| \\ &\qquad < (2N)^{-1} \rho_1^{N+2} \quad \text{in} \quad \rho_1 \leq \rho \leq 1. \end{aligned} \tag{2.03}$$

Now let Z be a function of ξ with continuous derivatives which equals zero for $\rho \leq \rho_1$ and equals 1 in the shell $\rho_2 \leq \rho \leq 1$, with $\rho_1 < \rho_2 < 1$. We may consider Z a function of x in $\mathfrak{R}$ with

$$Z = 0 \quad \text{in} \quad \mathfrak{R} - \mathfrak{S}^1, \quad Z = 1 \quad \text{in} \quad \mathfrak{S}^2, \tag{2.04}$$

$\mathcal{S}^1$ and $\mathcal{S}^2$ being the images of the shells $\rho_1 < \rho < 1$ and $\rho_2 \leq \rho \leq 1$. Then we set

$$\Omega_l = 2\, Z \Sigma_\mu \xi_\mu (1 - \rho^{-N}) X_{l/\mu}. \tag{2.05}$$

Clearly, these functions Ω_l have bounded continuous derivatives in $\mathcal{R}$ and vanish on $\mathcal{B}$. In $\mathcal{S}^2$ we have in particular

$$\begin{aligned} \Sigma_\lambda \Omega_{\lambda/\lambda} = {} & 2N \Sigma_{\kappa,\lambda,\mu} \xi_\mu \xi_\kappa \rho^{-N-2} \xi_{\kappa/\mu} (X_{\lambda/\mu} - x_{\lambda/\mu}) \\ & + (1 - \rho^{-N}) B, \end{aligned} \tag{2.06}$$

where B is a bounded function. By virtue of assumption (2.03) relation (2.06) yields

$$\Sigma_\lambda \Omega_{\lambda/\lambda} \geq N - (1 - \rho_1^{-N}) \mid B \mid.$$

Hence a value ρ_3 with $\rho_2 < \rho_3 < 1$ can be found such that $\Sigma_\lambda \Omega_{\lambda/\lambda} \geq 1$ in $\mathcal{S}^3$, the image of $\rho_3 \leq \rho \leq 1$. Clearly a value of v can be found such that $\mathcal{S}^3$ contains $\mathcal{R} - \mathcal{R}_v$.

To construct the function H^δ described in the introduction we introduce the shells $\mathcal{S}_\epsilon$ adjacent to the boundary $\mathcal{B}$ as images of the shells $1 - \epsilon \leq \rho \leq 1$ in the ξ-space. With

$$t^2 = \text{max. } \Sigma_{\lambda,\mu} x^2 x^2_{\lambda/\mu}, \text{ in } \rho_0 \leq \rho \leq 1, \tag{2.07}$$

and

$$\tau^{-2} = \text{max. } \Sigma_{\kappa,\lambda} \xi^2_{\kappa/\lambda}, \text{ in } \mathcal{S}_{1-\rho_0}$$

we state that

$$\mathcal{R} - \mathcal{R}_{\tau\epsilon} < \mathcal{S}_\epsilon < \mathcal{R} - \mathcal{R}_{\tau\epsilon}.$$

Let P and P_ϵ be the images of two points on $\rho = 1$ and $\rho = 1 - \epsilon$ on the same ray in the ξ-space. Then, by (2.07), clearly $\mid P - P_\epsilon \mid \leq t\epsilon$ which implies that $\mathcal{S}$ lies in $\mathcal{R} - \mathcal{R}_{\tau\epsilon}$. Similarly one finds $\mathcal{S}_\epsilon > \mathcal{R} - \mathcal{R}_{\tau\epsilon}$ from (2.08).

To a given $\delta > 0$ we now set $\delta' = \delta\tau/2t$. Then we have

$$\mathcal{R} - \mathcal{R}_{\delta'} < \mathcal{S}_{\delta'/\tau} < \mathcal{S}_{\delta/t} < \mathcal{R} - \mathcal{R}_\delta.$$

We then choose H as a function of ξ with continuous derivatives such that $H = 1$ for $1 - \rho \geq \rho/l$ while $H = 0$ for $\delta'/\tau = \delta/2t > 1 - \rho \geq 0$. We can do this in such a manner that the derivatives of H with respect to $[\xi]$ are in absolute value less than $3t/\delta$. When we now consider H a function of x it is

clear that $\mathrm{H} = 0$ in $\mathfrak{S}_{\delta'/\tau}$, hence in $\mathfrak{R} - \mathfrak{R}_\delta$, $\mathrm{H} = 1$ in $\mathfrak{R} - \mathfrak{S}_{\delta/t}$, hence in $\mathfrak{R}_\delta$, and that the derivatives of H with respect to x_1 are less than $3N\,ht/\delta$ when $h = \max |\, \xi_{k/l} \,|$ in $\mathfrak{S}^0$. Thus the function $\mathrm{H} = \mathrm{H}^\delta$ satisfies the specifications set up in the introduction.

We note that a domain is also an-Ω-domain if its boundary consists of a finite number of pieces each of the character described at the beginning of this section.

It is evident that the domains here considered are also special Ω-domains (see the end of the introduction); we need only set $\mathfrak{R}^{(\delta)} = \mathfrak{R} - \mathfrak{S}_{\delta/t}$. For these domains the same approximating functions $X(\xi)$ can be used: $X^{(\delta)}(\xi) = X((1-\epsilon)\xi)$. It is then obvious that the derivatives of the functions $\Omega^{(\delta)}$ are uniformly bounded.

We proceed to show that domains *with corners or edges* are also Ω-domains. We first consider the case that the domain $\mathfrak{R}$ has a corner or an edge at the origin O in such a way that, in a neighborhood of O, $\mathfrak{R}$ can be represented by

$$0 < x_1, 0 < x_2 \cdots, 0 < x_n; \text{ for } n \leq N.$$

We then must construct a vector Ω_k defined for $0 \leq x_1, \cdots, x_n$, which is continuous up to the boundary $x_1 x_2 \cdots x_n = 0$ and vanishes there, which in $0 \leq x_1, \cdots, x_n$ possesses continuous bounded derivatives, and for which

$$\Omega_{1/1} + \cdots + \Omega_{n/n} \geq 1. \tag{2.09}$$

Such a vector is

$$\begin{aligned} \Omega_1 = \cdots = \Omega_n &= n[x_1^{-1} + \cdots + x_n^{-1}]^{-1}, \\ &\quad \text{for } 0 < x_1, \cdots, x_n; \\ &= 0, \\ &\quad \text{for } x_1 x_2 \cdots x_n = 0. \end{aligned} \tag{2.10}$$

In other words all components are equal and just the harmonic mean of the coordinates. To verify that this vector has the desired properties we first note the relation

$$|\, \Omega_k \,| \leq n \min x_\nu, \qquad 0 < x_1, \cdots, x_n; \quad \nu = 1, \cdots, n;$$

which is immediately derived from the definition (2.10) and implies that $\Omega_k \to 0$ uniformly as x approaches the boundary $x_1 \cdots x_n = 0$. We further find

$$\Omega_{k/l} = nx_l^{-2}[x_1^{-1} \cdots + x_n^{-1}]^{-2}, \qquad l = 1, \cdots, n,$$
$$= 0 \qquad\qquad\qquad\qquad l > n,$$

whence $$0 \leq \Omega_{k/l} \leq n \text{ in } 0 < x_1, \cdots, x_n.$$

Finally, (2. 09) obtains,

$$\Omega_{1/1} + \cdots + \Omega_{n/n} = n[x_1^{-2} + \cdots + x_n^{-2}][x_1^{-1} + \cdots + x_n^{-1}]^{-2} \geq 1.$$

The existence of function H^δ with the desired properties is easily seen. Introducing a function $\eta(\xi)$ of a single variable ξ with continuous second derivatives for $0 < \xi$ and the properties $0 \leq \eta(\xi) \leq 1$

$$\eta(\xi) = 0 \quad \text{for } \xi \leq 1/2$$
$$= 1 \quad \text{for } \xi \geq 1,$$

we set

$$\mathrm{H}^\delta = \eta(x_1/\delta) \cdots \eta(x_n/\delta).$$

This function has evidently the desired properties with $\delta' = \delta/2$.

If the domain $\mathfrak{R}$ is a cube

$$0 < x_1, \cdots, x_N \leq 1$$

we define the vector $\Omega_k{}^{(m)}$ with $n = N$ for each of the 2^N corners $\mathfrak{k}^{(m)}$, $m = 1, \cdots, 2^N$. For the corner $\mathfrak{k}^{(1)} = 0$ we define $\zeta(x)$ as a function with continuous derivatives such that

$$0 \leq \zeta(\xi) \leq 1, \quad 0 \leq \xi \leq 1,$$
$$\zeta(\xi) = 1, \quad 0 \leq \xi \leq \tfrac{2}{3},$$
$$= 0, \quad \tfrac{5}{6} \leq \xi \leq 1.$$

We introduce the function $\mathrm{Z}^{(1)}(x)$ by

$$\mathrm{Z}^{(1)}(x) = \zeta(x_1) \cdots \zeta(x_N).$$

and define the functions $\mathrm{Z}^{(2)}(x), \cdots, \mathrm{Z}^{(2^N)}(x)$ correspondingly. Then we set

$$\Omega_1 = \cdots = \Omega_N = \mathrm{Z}^{(1)}\Omega^{(1)} + \cdots + \mathrm{Z}^{(2^N)}\Omega^{(2^N)}.$$

One immediately verifies that this vector has the desired properties. The

functions H^δ for the cube are constructed from those for a single corner in a similar manner. Thus it is seen that *a cube is an* Ω-domain. Evidently a cube is also a special Ω-domain.

Suppose that the corner in *not rectangular* as assumed so far but that it can be obtained from a rectangular corner by a transformation $x_l = x_l(x^*)$ with positive jacobian and bounded second derivatives. Then one sets, in obvious notation

$$\Omega_k = J_0 J^{-1} \Sigma_\lambda \Omega^*{}_\lambda \partial x_k / \partial x_\lambda$$

where J_0 is the minimum of J in the neighborhood considered. By a well known invariance property of the divergence

$$\Sigma_\kappa \partial \Omega_\kappa / \partial x_\kappa = J_0 J^{-1} \Sigma_\lambda \partial \Omega^*{}_\lambda / \partial x^*{}_\lambda \geq J_0 J^{-1} \leq 1,$$

and hence Ω_k enjoys the desired properties. The construction of the functions H^δ is obvious. Thus it is shown that also certain domains with non-rectangular corners or edges are Ω-domains, for which our inequality holds.

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TOPOLOGIES FOR HOMEOMORPHISM GROUPS.*

By RICHARD ARENS.

1. Introduction. This paper investigates the topological space obtained by defining, in a group H of homeomorphisms [1] of a locally compact space A, one or the other of the following two topologies:

a) The k-topology for H is based on neighborhoods U of the following description: Let K and W be compact and open subsets, respectively, of A, and let (K, W) denote the collection of all elements of H which transform K onto a part of W. Let U be the intersection in H of any finite number of all possible such sets, (K, W).

b) The g-topology for H is based on neighborhoods U of the following nature: Let K and W be closed and open subsets, respectively, of A, with the condition that either K, or the complement of W in A, is compact. Let (K, W) contain all those elements of H which send K into W, and let U be the intersection of an arbitrary finite set of such sets as (K, W).

With the k-topology, the binary group operation [2] in H becomes continuous. With the g-topology, we have moreover a continuous group operation of inversion whence H forms a topological group [3] with the g-topology.

In fact, the g-topology is the strongest [4] topology which makes H into a topological group and at the same time insures that $h(x)$ depends continuously on x and h simultaneously.

The k- and g-topologies coincide if A is compact. If A is also metric, these topologies coincide with that studied, for example, by G. Birkhoff [3, p. 872] for the group H of homeomorphisms of a compact metric space,

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[1] For this, and other general topological terms, we follow the usage of Lefschetz (reference [8] in the list at the end of this paper).

[2] In which, for f, g, h, in H, one has $f = gh$ if and only if, for each x in A, $f(x) = g(h(x))$.

[3] Which is not usually complete (in the sense of Weil, [15, p. 31]) but satisfies a condition (see 7) which insures that H is of the second category [2, p. 13] at least if A is perfectly separable.

[4] I. e., providing the most limit elements.

where one metrizes H by setting (f, g) equal to the maximum of the distance from $f(x)$ to $g(x)$, for all x in A.

van der Waerden and van Dantzig [14, p. 370] apply this method to the locally compact, separable metric case by first adding a point to A to make it compact, and metrizing as above. This produces the g-topology. We show that, in general, the g-topologized group H of homeomorphisms of A is topologically isomorphic to the k-topologized group H^* of homeomorphisms of A^* leaving I fixed, where $A^* = A + I$ is the compactification [8, p. 23] of A.

Another result is that if h_ν is a directed set of homeomorphisms of a locally compact, *locally connected* space, which converges to the identity in the k-topology, then the directed set h_ν^{-1} also converges to the identity, in the k-topology. Thus, for homeomorphism groups of locally compact, locally connected spaces, the g- and k-topologies are equivalent.

If H is a compact and effective transformation group (Cf. [9, p. 365]) its topology must coincide with the k- and g-topologies; this is true in particular for compact Lie groups of transformations. For locally compact Lie groups of transformations, the maximal connected subsets of neighborhoods in the g-topology provide a basis for the usual parameter-space topology.

Directed sets of functions, f_ν, defined on a locally compact Hausdorff space, have the following convergence properties involving the k- and g-topologies: [5]

i) f_ν converges to f in the k-topology if and only if $f_\nu(x_\delta)$ converges to $f(x)$ whenever the directed set x_δ converges to x in the domain.

ii) f_ν converges to f in the g-topology if and only if f_ν converges to f, and f_ν^{-1} converges to f^{-1}, in the k-topology, for homeomorphisms.

iii) f_ν converges to f in the g-topology if and only if the convergence of x_δ to x implies and follows from the convergence of $f_\nu(x_\delta)$ to $f(x)$.

We generalize a result of van Dantzig and van der Waerden in showing that an equicontinuous group [6] of homeomorphisms of a connected locally compact space is a locally compact topological group.

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[5] Cf. a separate and more complete investigation of the k-topology in Arens [1]. See also [14].

[6] I. e., a group whose elements as a whole form an equicontinuous family in the usual metric sense, or in a more general sense expressible by means of uniform structure (Cf. 7 below). For the way in which compactness properties of the group imply equicontinuity, see Eilenberg [6, p. 77].

We formulate a condition upon a topological group of homeomorphisms which insures that the quotient space H/F, where F is the subgroup of elements leaving a point x of A fixed, should be homeomorphic to the domain of transitivity (under H) which contains x: If U be any open set in H, then the set of all $h(x)$, where $h \in U$, is open relative to its domain of transitivity in A. This condition is satisfied if H is compact;[7] and we prove it true also for any transitive Lie group of transformations.[8]

2. Definition and formal interrelation of *k*- and *g*-topologies. Let A be any topological space, and let C be any class of continuous functions defined on A and with values also in A. If K and W are subsets of A, we shall always use (K, W) to denote the collection of functions $f \in C$ for which $f(K) \subset W$. If $K_1, W_1, \cdots, K_n, W_n$, where n is finite, are n pairs of subsets of A, we shall denote by $(K_1, \cdots, K_n; W_1, \cdots, W_n)$ the set of all functions $f \in C$ for which $f(K_i) \subset W_i$, $i = 1, 2, \cdots, n$.

DEFINITION. The topology obtained in C by taking the collection of all the sets $(K_1, \cdots, K_n; W_1, \cdots, W_n)$ as neighborhoods of any functions they contain,[9] where the K's are compact and the W's are open, will be called the k-topology for C.[10]

DEFINITION. The topology obtained in C by taking the collection of all the sets $(K_1, \cdots, K_n; W_1, \cdots, W_n)$ as neighborhoods of any functions they contain,[9] where the K's are closed and the W's are open, and, for each i, either K_i or the complement W'_i of W_i is compact, will be called the g-topology for C.

We may adjoin a point I to A, giving $A^* = A + I$, which is compact, by defining the neighborhoods of I to be the exteriors of compact sets of A [8, p. 23]. We may then regard any function $f \in C$ as corresponding to a function f^* defined on A^*, but mapping I, and only I, on I. We thus arrive at a class C^* of continuous functions each mapping A^* into itself, which is in obvious $1-1$ correspondence with C.

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[7] Cf. Montgomery and Samelson [10, p. 455].

[8] We give an example of a compact subset of 3-space for whose group of homeomorphisms H the condition fails for every topology for H.

[9] Or, equivalently, taking the sets (K, W) as a subbase [8, p. 6]. In proofs, whenever it involves no loss in generality, we shall tacitly limit ourselves to the consideration of subbasic neighborhoods.

[10] This topology has been defined, in one way or another, for various special cases by many authors, such as Banach [2, p. 11], G. Birkhoff, *loc. cit.*, Fox [7, p. 429], Pontrjagin [12, p. 127].

LEMMA 1. *C with the g-topology, and C^* with the k-topology are homeomorphic under the obvious correspondence.*[11]

The proof of this is rather obvious, and will be omitted.

From this we can infer at once:

THEOREM 1. *The g-topologized group H of homeomorphisms of a locally compact Hausdorff space A is topologically isomorphic to the group H^* of those homeomorphisms of the compactification A^* of A which leave the ideal point fixed.*[12]

Of course, it is evident from the definition that the k- and g-topologies coincide when A is compact.

3. A minimal property of the k-topology.

DEFINITION. A topology for C (cf. **2**) will be called *admissible,* if with $x \in A$ and $f \in C$, $F(f, x) = f(x)$ is a continuous function on $C \times A$ into A.

This is equivalent to saying that whenever $f \in C$ and $x \in A$, and a neighborhood W of $f(x)$ are given, we can find a neighborhood $V(x)$ and a neighborhood $U(f)$ such that each function in U maps any point of V onto a point of W.

DEFINITION. If t and t^* are two topologies for the same set of elements, and if every open set of t is an open set of t^*, we shall say that t is *stronger* than t^*, and shall write $t \subset t^*$.

THEOREM 2. *If A is a locally compact Hausdorff space, and C is a class of continuous functions defined on A, then the k-topology is the strongest admissible topology that can be given to C.*

We refer the reader to [1, p. 482] for a proof of Theorem 2.

In particular, we can say that the group of homeomorphisms of a locally compact Hausdorff space can be given a strongest admissible topology.

To show the rôle of local compactness, we prove the following:

Given any admissible topology for the group of homeomorphisms H of the rational number system, one can construct another admissible topology for H which is not weaker than the first.

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[11] This correspondence is also an isomorphism if C, and thus C^*, are groups of homeomorphisms.

[12] This proposition was suggested to us by G. Birkhoff.

Proof. Let H have an admissible topology. Then one can certainly find a neighborhood U of the identical homeomorphism and an interval V about 0 such that for $h \in U$ and $x \in V$ one will have $|h(x)| < 1$. Now let ξ be an irrational number lying between the endpoints of V. Construct a new topology for H by employing as subbasic neighborhoods sets of the form (F, W) where F is closed and contains no sequence (of rational numbers) converging to ξ, and W is open. This topology is clearly admissible. We shall show that none of the new neighborhoods of the identity, say $U^* = (F_1, \cdots, F_n; W_1, \cdots, W_n)$, is contained in U. For then surely we can find an open-closed interval surrounding ξ, and contained within V, which does not meet $F_1 \cup \cdots \cup F_n$; and a homeomorphism h can be set up which interchanges this open-closed interval with a corresponding neighborhood of $10 + \xi$, but leaves all other points fixed. Then $h \in U^*$ but h certainly does not lie in U.

4. Topologizing homeomorphism groups.

THEOREM 3. *The group of homeomorphisms of a locally compact Hausdorff space, becomes, with the g-topology, a topological group; and the g-topology is the strongest admissible topology for which this is true.*

Proof. We consider first the group H of homeomorphisms of a compact space A^*. In this case, the g- and k-topologies coincide, and the latter has just been proved to be the strongest admissible topology. Now let (K, W) be a neighborhood of a homeomorphism h. This means that $h(K) \subset W$, or that $h(K') \supset W'$, or that $h^{-1}(W') \subset K'$, whence (W', K') is a neighborhood of h^{-1}. Thus the process of set complementation in A^* shows that group-inversion transforms neighborhoods into neighborhoods, in H, and thus that the passage from h to h^{-1} is continuous.

Now let f and g be homeomorphisms and let a neighborhood of $h = fg$ (see [2]), (K, W), be given. This means that $f(g(K)) \subset W$, i.e., that $g(K) \subset f^{-1}(W)$. Since A^* is normal, we can find an open set V such that $g(K) \subset V$, $V^- \subset f^{-1}(W)$. Therefore (K, V) and (V^-, W) are neighborhoods of g and f, respectively, and it is easy to see that if g' belongs to the former, and f' to the latter, then $f'g'$ will send K into W, or $f'g' \in (K, W)$. Thus the group operations in H are all continuous: H is a topological group.

If H is the group of homeomorphisms of a locally compact Hausdorff space A, then let us apply Theorem 1 and consider the corresponding group H^* of homeomorphisms leaving the ideal point $I = A^* \cap A'$ fixed. Now H^* is clearly a closed subgroup of the group of all homeomorphisms of A^*, for if, for some h, $h(I) \neq I$, then (I, A) is a neighborhood of h not meeting H^*. There-

fore, by the preceding case, H^* is a topological group, and thus so also is H, with the g-topology.

Now suppose that we have another topology t^* for H, which is admissible and makes the passage h to h^{-1} bicontinuous in H. Let a g-neighborhood $U = (K, W)$ be given. If this is not already a k-neighborhood, $U^{-1} = (W', K')$ will be, and thus either U or U^{-1}, and hence both, will be open in the topology t^*, since the k-topology is the strongest admissible one. Therefore the topology t^* is weaker than the g-topology also.

REMARK. If in the definition of k-topology, we merely require that the set K be closed, we obtain a topology which makes the group of homeomorphisms of any normal space into a topological group. The proof resembles that of Theorem 3.

5. Homeomorphisms of locally connected spaces.

DEFINITION. A *locally connected topological space* is one in which the connected, open sets form a *complete system* of neighborhoods.

THEOREM 4. *If H is the group of homeomorphisms of a locally connected, locally compact Hausdorff space, and H is given the k-topology, then h^{-1} varies continuously with h in H, and H forms a topological group.*

Proof. The binary group operation will first be shown to be continuous; the local connectedness of the space A is not needed for this. Let (K, W) be a neighborhood of fg. This means that $g(K) \subset f^{-1}(W)$. Since A is locally compact, we can, at each point $x \in g(K)$, find an open set $V(x)$ whose closure is compact and contained in $f^{-1}(W)$. This covering by the V's can be reduced to a finite cover, and we have:

$$g(K) \subset V \subset V^- \subset f^{-1}(W), \text{ where } V = V(x_1) \cup \cdots \cup V(x_n),\ x_1, \cdots, x_n \in g(K)$$

Therefore (K, V) and (V^-, W) are neighborhoods of g and f, respectively, and if g' belongs to the former, and f' to the latter, then (as in Theorem 3) $f'g'$ must lie in (K, W).

Before proceeding to a discussion of h^{-1}, let us establish that the sets of the form (L, W), where L is compact, connected, and has a non-void interior, and W is open, constitute a subbase for the k-topology of H. Let $h \in (K, W)$ where K is any compact set. As W is locally connected, we can, for each $x \in K$, find a connected open set $V(x)$, whose closure is compact, such that $f(V(x)^-) \subset W$. Then $x_1, \cdots, x_n$ can be found such that $K \subset L_1 \cup \cdots \cup L_n$ where $L_i = V(x_i)^-$, which is connected. Clearly $(L_1, \cdots, L_n; W_1, \cdots, W_n) \subset (K, W)$, as desired.

Now let (L, W) be a k-neighborhood of f^{-1}, where L is compact, connected, and has a non-void interior $L'^{-\prime}$. As demonstrated in the earlier part of this proof, we can find an open set G such that $f^{-1}(L) \subset G \subset G^- \subset W$, where G^- is compact. We may similarly select an open set V containing G^-, with V^- compact and included in W. Then $f(G' \cap V^-) \subset L' \cap f(W)$. We can also find a point x for which $f(x) \in L'^{-\prime}$. Therefore $(x, G' \cap V^-; L'^{-\prime}, L' \cap f(W))$ is a k-neighborhood of f, which we shall refer to as U_0.

If $h \in U_0$, then $h(G' \cap V^-) \subset L' \cap f(W)$. By taking complements and reversing the inclusion sign, $L \cup f(W)' \subset h(G \cup V^{-\prime})$. Since $h(G)$ and $h(V^{-\prime})$ are disjoint open sets, and L is connected, L must lie in the one or the other. It cannot lie in $h(V^{-\prime})$ because $h(x) \in L'^{-\prime}$ and x is not in $V^{-\prime}$. Hence $L \subset h(G)$, or $h^{-1}(L) \subset G \subset W$. But this means that $h^{-1} \in (L, W)$. Thus inversion is continuous in H when it has the k-topology.

In other words, H forms a topological group.

6. Convergence in the k- and g-topologies.

DEFINITION. A directed system Δ is a partially ordered class with the property that if δ_1 and δ_2 are elements of Δ, then there is a $\delta \in \Delta$ for which $\delta > \delta_1$, and $\delta > \delta_2$.

DEFINITION. A function $\bar{x}$ defined on a directed system and taking its values in a topological space A is called a directed set. The value of $\bar{x}$ at $\delta \in \Delta$ is usally written x_δ and not $\bar{x}(\delta)$. When it causes no confusion, we shall call $\bar{x}$ "the directed set x_δ." x_δ will be said to converge to a limit $x \in A$ (in symbols, $x_\delta \to x$ or $\lim_{\delta \in \Delta} x_\delta = \lim_\delta x_\delta = x$) when for every neighborhood V of x, there is a $\delta_0 \in \Delta$ such that $\delta > \delta_0$ implies $x_\delta \in V$.

DEFINITION. Let Δ and $\mathbf{N}$, and therefore $\Sigma = \Delta \times \mathbf{N}$, be directed systems.[13] Let x_δ be a directed set, and suppose that for each $\sigma = (\delta, \nu)$, $x'_\sigma = x_{\delta'}$ where $\delta' > \delta$. Then we shall call x'_σ a directed subset of x_δ. The directed subset x'_σ will sometimes be denoted by the symbol $x_{\delta'}$.

The utility of this definition lies in the following lemma, which we shall occasionally use.

LEMMA 2. *If $x_\delta \to x$ and $x_{\delta'}$ is a directed subset of x_δ, then $x_{\delta'} \to x$. If x_δ is a directed set in a compact space A, then some directed subset of x_δ converges.*

Proof. Suppose that $x_\delta \to x$. Now, if a neighborhood $V(x)$ is given, then, for some δ_0, $\delta > \delta_0$ implies $x_\delta \in V$. Let $\sigma = (\delta, \nu)$ with any $\nu \in \mathbf{N}$. Then

[13] If $\sigma \in \Sigma$, then $\sigma = (\delta, \nu)$; $(\delta, \nu) > (\delta_0, \nu_0)$ means $\delta > \delta_0$ and $\nu > \nu_0$.

$\sigma > \sigma_0$ implies $x'_\sigma \in V$, whence $x_{\delta'} \to x$. Now suppose that $x_\delta \in A$, A being compact. Suppose that for each $x \in A$, there is a neighborhood $V(x)$ and a δ_x such that if $\delta > \delta_x$ then x_δ cannot lie in $V(x)$. Suppose that $A = V(x) \cup \cdots \cup V(z)$. Let $\delta > \delta_x, \cdots, \delta_z$. Then x_δ cannot lie in A. This contradiction shows that for some $x \in A$, and any δ and neighborhood V of x, then for some $\delta' > \delta$, $x_{\delta'} \in V$. The neighborhoods of x forming a directed system, we have $x_{\delta'} \to x$.

We can now consider convergence [14] properties of the k- and g-topologies.

THEOREM 5. *Let h_ν be a directed set in the group H of homeomorphisms of a locally compact Hausdorff space A. Then,*

i) *$h_\nu \to h$ in the k-topology if and only if $h_\nu(x_\delta) \to h(x)$ whenever $x_\delta \to x$ in A.*

ii) *$h_\nu \to h$ in the g-topology if and only if $h_\nu \to h$ and $h_\nu^{-1} \to h^{-1}$ in the k-topology.*

iii) *$h_\nu \to h$ in the g-topology if and only if $x_\delta \to x$ is always equivalent to $h_\nu(x_\delta) \to h(x)$ for any directed set x_δ in A.*

Proof. If $h_\nu \to h$ in the k-topology, then $h_\nu(x_\delta) \to h(x)$ whenever $x_\delta \to x$, because the k-topology is admissible. And if $h_\nu(x_\delta) \to h(x)$ whenever $x_\delta \to x$, one may introduce an admissible topology into the set $\{h, h_\nu\} = D$ by taking as the μ-th neighborhood of h, the set U_μ of all h_ν with $\nu > \mu$, and h. All $h_\nu \neq h$ shall be considered isolated. This patently gives D an admissible topology, and one in which $h_\nu \to h$. Since the k-topology is stronger than any admissible topology, $h_\nu \to h$ in the k-topology. This proves i).

For ii), suppose $h_\nu \to h$ and $h_\nu^{-1} \to h^{-1}$ in the k-topology. Let any subbasic neighborhood of h in the g-topology be given, say (K, W). If this is not already a k-neighborhood, then (W', K') must be a k-neighborhood of h^{-1}. Hence, for some ν_0, $\nu > \nu_0$ will imply $h_\nu^{-1}(W') \subset K'$, or $h_\nu(K) \subset W$, whence $h_\nu \to h$ in the g-topology. If (K, W) is a k-neighborhood, then for some ν_0, $\nu > \nu_0$ will imply $h_\nu(K) \subset W$, with the same result. If h_ν converges in the g-topology, it certainly converges in the k-topology. Thus we have ii).

iii) can be obtained by combining i) and ii).

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[14] One of the most important properties, but one which we shall not require, is that for any directed set f_ν of continuous functions $f_\nu \to f$ in the k-topology if and only if $f_\nu(x)$ converges uniformly, on compact sets, to $f(x)$ [1]. The range space, of course, must be a metric, or at least a uniform, space.

An immediate application of this is a reformulation of the essential part of Theorem 4:

THEOREM 4′. *If A is a locally compact, locally connected Hausdorff space, and if h_ν is a directed set of homeomorphisms of A having the property that $h_\nu(x_\delta) \to x$ whenever $x_\delta \to x$, then $h_\nu^{-1}(x_\delta) \to x$ whenever $x_\delta \to x$.*

We shall give an example showing that local connectedness is not redundant here (nor in Theorem 4):

Let A be the set of real numbers $0, 1, 2, \cdots$ and $1/2, 1/3, \cdots$. Then A is locally compact, but not locally connected at 0. Consider the sequence $h_1, h_2, \cdots$ of homeomorphisms of A where

$$h_k(m) = \begin{cases} m \text{ if } m = 0, 1, 2, \cdots, k-1 \\ 1/k \text{ if } m = k \\ m-1 \text{ if } m = k+1, k+2, \cdots \end{cases}$$

$$h_k(1/m) = \begin{cases} 1/m \text{ if } m = 2, 3, \cdots, k-1 \\ 1/m+1 \text{ if } m = k, k+1, \cdots. \end{cases}$$

Since $1/m$ is the only converging sequence in A, we have only to note that $h_k(1/m) \to 0$ to see that h_k converges to the identity in the k-topology. But $h_k^{-1}(1/k) = k$, which diverges to ∞.

This shows also that the k-topology does not generally coincide with the g-topology, for h_k evidently does not converge in the latter.

7. Uniform structures in Spaces and Groups. Let A be any space, and let, as usual, $A \times A$ denote the class of all pairs (x, y), where $x, y \in A$. Let I denote the class of all pairs (x, x).

DEFINITION. A family $\mathfrak{A}$ of subsets of $A \times A$ is called a uniform structure for A if [Bourbaki 4, pp. 85, 86, and 92]

I. Each $R \in \mathfrak{A}$ contains I as a subset, and if $x \neq y$, then some $R \in \mathfrak{A}$ does not contain (x, y).

II. If R and S belong to $\mathfrak{A}$, then so does $R \cap S$.

III. If R and S belong to $\mathfrak{A}$, then there is an $S \in \mathfrak{A}$ such that $SS \subset R$.[15]

[15] By RS we mean the class of all pairs (x, z) for which there is an element $y \in A$ such that $(x, y) \in R$ and $(y, z) \in S$. We define S^n, inductively, $S^n = S(S^{n-1})$ and $S^2 = SS$. By $S(x)$ we mean the subset of A consisting of all z such that $(x, z) \in S \in \mathfrak{A}$. By R^{-1} we mean the class of all pairs in R, with the order reversed.

IV. $R^{-1} = R$, and the sets $R(x)$ form a complete system of neighborhoods for the point x in A.

For example, given any topological group H, one can define a uniform structure $\mathfrak{H}^*$ by saying that $(x, y) \in U^*$ if $xy^{-1} \in U$, and U is a neighborhood of the identity in H. Another uniform structure $^*\mathfrak{H}$ can be obtained by setting $(x, y) \in {}^*U$ if $x^{-1}y \in U$, U being any neighborhood of the identity in H. One may define a third uniform structure $^*\mathfrak{H}^*$ in which the relations are ${}^*U^* = {}^*U \cap U^*$, U^* and ${}^*U$ being defined as above [Weil, 15, p. 30].

A topological space A on which a uniform structure has been defined will be called a uniform space.

DEFINITION. Let A have a uniform structure $\mathfrak{A}$, and let x_δ be a directed set in A such that for each $R \in \mathfrak{A}$ there is a δ_0 such that $\delta, \delta_1 > \delta_0$ implies $(x_\delta, x_{\delta_1}) \in R$. Then x_δ will be called a Cauchy directed set (relative to $\mathfrak{A}$). If every Cauchy directed set in A converges (has a limit in A), then A will be called complete (in the structure $\mathfrak{A}$).

A topological group is conventionally called complete if it is complete in $\mathfrak{H}^*$ or $^*\mathfrak{H}$, for if it is complete in one, it is likewise in the other (Weil, *loc. cit.*).

THEOREM 6. *The g-topologized group H of homeomorphisms of a locally compact Hausdorff space is always complete in $^*\mathfrak{H}^*$, but not generally complete in $\mathfrak{H}^*$ and $^*\mathfrak{H}$.*

Proof. It will suffice to consider the k-topologized group of homeomorphisms of a compact Hausdorff space A, by Theorem 1, and the fact that the subgroup leaving a certain point fixed is always closed.

Suppose that h_ν is a Cauchy directed set relative to $^*\mathfrak{H}^*$. This means that $h_\nu^{-1}h_\mu \to$ identity and $h_\nu h_\mu^{-1} \to$ identity. For any $x \in A$, since A is compact, $h_\nu(x)$ will have a convergent directed subset $h_{\nu'}(x)$. Combining $h_{\nu'}(x) \to y$ with $h_\nu h_{\nu'}^{-1} \to$ identity by Theorem 5, i), we obtain the result that $h_\nu(x)$ converges for each x. Let its limit be called $h(x)$. We shall show that $h_\nu \to h$ in the k-topology. Suppose that $x_\delta \to x$. If $h_\nu(x_\delta) \nrightarrow h(x)$, using the compactness of A, suppose that $h_{\nu'}(x_{\delta'}) \to y \neq h(x)$. By the previous reasoning, we have:

$$\lim_{\mu,\nu,\delta} h_\mu h_{\nu'}^{-1} h_{\nu'}(x_{\delta'}) = \lim_{\mu,\delta} h_\mu(x_{\delta'}) = y.$$

Now

$$\lim_\delta h_\mu(x_{\delta'}) = h_\mu(x) \text{ and } h_\mu(x) \to h(x).$$

Then

$$\lim_\mu \lim_\delta h_\mu(x_{\delta'}) = \lim_{\mu,\delta} h_\mu(x_{\delta'})$$

since the inner limit on the left, and the double limit on the right have both been shown to exist [Cf. Bourbaki, 4, p. 49]. This leads to $y = h(x)$, and this contradiction proves $h_\nu(x_\delta) \to h(x)$. Thus by Theorem 5, i), $h_\nu \to h$ in the k-topology. Let us also show that h is continuous. We now know that if $x_\delta \to x$, then $h_\nu(x_\delta) \to h(x)$. Therefore this double limit must equal the iterated limit

$$\lim_\delta \lim_\nu h_\nu(x_\delta) = \lim_\delta h(x_\delta)$$

since the inner limit exists, and hence $h(x_\delta) \to h(x)$. Because the condition on h_ν is equivalent to the same condition on h_ν^{-1}, we can conclude that $h_\nu^{-1}(x)$ also converges to a unique limit for each x, and defines a continuous function, $h^*(x)$. Applying Theorem 5, i) again, we have $h_\mu h_\nu^{-1}(x) \to h(h^*(x))$. But since, by hypothesis, $h_\mu^{-1} h_\nu \to$ identity, we can conclude $h(h^*(x)) = x$. Thus $h \in H$. This proves that H is complete in the structure $^*\mathfrak{H}^*$.

To show that H is not generally complete in $\mathfrak{H}^*$, consider the sequence h_n of homeomorphisms of the closed interval $[0, 2] = A$ where

$$h_n(x) = \begin{cases} tx, & 0 \leq x \leq 1 \\ (2-t)(x-1) + t, & 1 \leq x \leq 2, \end{cases} \text{ and } t = 1/n.$$

One can verify easily that $h_n h_m^{-1}$ converges uniformly to the identity, but $h_n(x) \to 0$ for each x in $[0, 1]$.[16]

Remarks on the compact metric case: If A has a metric, m, then the uniform structure $\mathfrak{H}^*$ corresponds to the well known metric m_1 in H, where $m_1(h, g) = \sup m(h(x), g(x))$, $x \in A$ [Cf. 3, p. 872], while $^*\mathfrak{H}^*$ corresponds to the metric m_2 where $m_2(h, g)$ is the larger of $m_1(h, g)$ and $m_1(h^{-1}, g^{-1})$, used by Oxtoby and Ulam [15, p. 879]. These metrics are of course equivalent (Cf. [15], *loc. cit.*) (perhaps this can be seen most quickly by noting that both determine the k-topology). This shows that although H is not complete in the sense of van Dantzig [5, p. 612], i. e., in m_1, it is of the second category, since it is complete in m_2.

8. Equicontinuous groups of homeomorphisms. The following is an obvious generalization of the concept of groups of isometries of a metric space.

DEFINITION. An equicontinuous group H of homeomorphisms of a topological space A with a uniform structure $\mathfrak{A}$ is one such that for each $R \in \mathfrak{A}$ there is an $S \in \mathfrak{A}$ such that $(x, y) \in S$ implies $(h(x), h(y)) \in R$ for any $h \in H$ and any x, y in A.

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[16] By using the remark of [14], one may prove in general that if $h_\nu \to f$, where f is continuous, then $h_\nu h_\mu^{-1} \to$ identity, in the k-topology.

LEMMA 3. *If H is an equicontinuous group of homeomorphisms of a uniform space A, then A can be given a uniformly equivalent* [15, p. 9] *structure $\mathfrak{A}^*$, which is invariant under H.*

Proof. For every $R \in \mathfrak{A}$, let R^* be the class of all pairs (u, v) such that for some $h \in H$ and some $(x, y) \in R$, $u = h(x)$ and $v = h(y)$. Since H has an identity element, $R \subset R^*$. But as H is equicontinuous, there is an S such that $(x, y) \in S$ implies $(h(x), h(y)) \in R$ for every h in H, whence $S^* \subset R \subset R^*$. This makes $\mathfrak{A}^*$ uniformly equivalent to $\mathfrak{A}$. If $(u, v) \in R^*$, then, for some x, y, h, we have $(x, y) \in R$ and $u = h(x)$ and $v = h(y)$, and so for any $g \in H$, $(g(u), g(v)) = (gh(x), gh(y)) \in R^*$. Therefore H sends R^* into R^*, and as H is a group, $\mathfrak{A}^*$ must be invariant.

COROLLARY. *If H is an equicontinuous group of homeomorphisms of a metric space A, then A can be given a uniformly equivalent metric which makes H a group of isometries.*[17]

This follows from the fact that a countable uniform structure determines a uniformly equivalent metric, and conversely [Weil, *op. cit.*, p. 16].

The following is a generalization of a theorem of van der Waerden and van Dantzig [14, p. 374].

THEOREM 7. *Any equicontinuous group H of homeomorphisms with the g-topology, of a locally compact, connected uniform space A is dense in some (equicontinuous) locally compact topological group H^- of homeomorphisms of A. In H^-, h_ν converges in the g-topology if and only if $h_\nu(x)$ converges for each x in A.*

Proof. Consider the set H^- of all functions on A into itself which are pointwise limits of directed sets in H. We shall prove that H^- has the properties set forth. Let us show first that H^- contains only homeomorphisms.

Suppose that $f \in H^-$; then some directed set h_ν converges pointwise to f. Suppose that $R \in \mathfrak{A}$, and that $S \in \mathfrak{A}$ is such that $(x, y) \in S$ implies $(h(x), h(y)) \in R$, and suppose that $(x, y) \in S$. Hence $(h_\nu(x), h_\nu(y)) \in R$ and so in the limit $(f(x), f(y)) \in R^3$ (see [15]). This shows that the whole set H^- is equicontinuous, and in particular, that f is continuous.

Given a relation R as above, and using the same $S \in \mathfrak{A}$, we can find a ν_0 such that $\nu > \nu_0$ implies $(h_\nu(x), f(x)) \in S$, and hence that $(x, h_\nu^{-1} f(x)) \in R$. Thus f has an inverse f^{-1} to which h_ν^{-1} converges pointwise, and which is therefore continuous. We need therefore only to prove that $f(A) = A$, and for

[17] Cf. Eilenberg, *op. cit.*, p. 79, § 8.

this we interpolate a lemma. In it we suppose that $\mathfrak{A}$ is invariant under H, which can always be done by Lemma 3.

LEMMA 4. *If $h_\nu \in H$ is a directed set, and $a \in A$, and $R^4(a)^-$ is compact, and if $h_\nu(b)$ converges for some $b \in R(a)$, then for each $x \in R(a)$, $h_\nu(x)$ has a convergent directed subset $h_{\nu'}(x)$.*

Proof. Select μ such that if $\nu > \mu$ then $(h_\nu(b), h_\mu(b)) \in R$. If $(x, a) \in R$, then $(h_\nu(x), h_\nu(a)) \in R$. This is true in particular if $x = b$ and also then for $\nu = \mu$. This gives four conditions which can be combined to $(h_\nu(x), h_\mu(a)) \in R^4$, or $h_\nu(x) \in R^4(h_\mu(a)) = h_\mu(R^4(a))$; and this last set has a compact closure because it is homeomorphic to $R^4(a)$. Hence $h_\nu(x)$ has a convergent directed subset, and Lemma 4 is proved.

Now $h_\nu^{-1}(b)$ converges at some point $b \in f(A)$. Let R, x, and a have the meaning they have in Lemma 4. Such an R exists for a given a because A is locally compact. Then $h_{\nu'}^{-1}(x)$ has a convergent directed subset which converges to a point, say $h_{\nu'}^{-1}(x) \to c$.

Let $S \in \mathfrak{A}$ be given, and select $T \in \mathfrak{A}$ such that $T^3 \subset S$. We can find a μ such that $\nu > \mu$ will imply $(h_\nu(c), f(c)) \in T$, and imply $(h_{\nu'}(c), h_\nu(c)) \in T$, and imply $(h_{\nu'}^{-1}(x), c) \in T$ or $(x, h_{\nu'}(c)) \in T$. This gives $(f(c), x) \in S$, and therefore $f(c) = x$.

Therefore x as well as b lies in $f(A)$, which is therefore open. On the other hand, the point a might have been taken as any point in $f(A)^-$, and x taken as coinciding with a. Thus $f(A)$ is also closed. Since it is not void, and since A is connected, $f(A) = A$.

Thus G^- contains only homeomorphisms.

Next we prove that convergence in the g-topology follows from pointwise convergence in H^-. We already know that if $h_\nu \to h$ pointwise, then $h_\nu^{-1} \to h^{-1}$ pointwise. Therefore we need only check that $h_\nu \to h$ in the k-topology, by Theorem 5.

Suppose that $x_\delta \to x$, and let $R \in \mathfrak{A}$ be given. Select $S \in \mathfrak{A}$ such that $S^3 \subset R$, and ν_0 and δ_0 such that $\nu > \nu_0$, $\delta > \delta_0$ implies $(x_\delta, x) \in S$ and $(h_\nu(x), h(x)) \in S$. As we have seen before, we can be sure that $(h_\nu(x_\delta), h_\nu(x)) \in S^3$ when $(x_\delta, x) \in S$, and so $(h_\nu(x_\delta), h(x)) \in R$, proving $h_\nu \to h$ in the k-topology.

Therefore H^- is a closed subgroup of the g-topologized group of all homeomorphisms of A, and thus is a topological group.

It remains to show that H^- is locally compact. Let $f \in H^-$ and select any point $a \in A$. Suppose that $R(f(x))^-$ is compact. We shall show that the neighborhood $U = (a, R(f(a)))$ of f has a compact closure. If h_ν is a

directed set in U, then $h_\nu(a)$ is a directed set in $R(f(a))$, and hence has a convergent directed subset, $h_{\nu'}(x)$. The argument following the proof of Lemma 4 may be applied to show that $h_{\nu'}(x)$ converges for every x. Thus U has a compact closure.

REMARK. H^- is easily seen to be compact if A is.

COROLLARY. (van der Waerden and van Dantzig, [14, p. 374]).[18] *The group of isometries of a locally compact metric space which contains but a finite number of connected components forms a locally compact topological group.*

9. Quotient Spaces. In this section, we shall develop some topological connections between groups of homeomorphisms and the regions over which they are transitive.

Suppose that H is a group of homeomorphisms of a topological space A, that b is a fixed point of A, that ${}_bH$ is the subgroup of H consisting of homeomorphisms which leave b invariant, and finally that $H(b)$ is the set of all points into which H can transform b,—otherwise known as the orbit of b (under H). Then there is a natural $1-1$ correspondence between cosets of ${}_bH$ and the points of $H(b)$ (Cf. van Dantzig [5, p. 610]),

$$\phi : \phi({}_bHg) = g(b).$$

This correspondence is possible because $g(b)$ depends only on the coset of ${}_bH$ in H in which the homeomorphism g lies.

LEMMA 5. *If a group H of homeomorphisms of a topological space A has an admissible topology, and the natural topology* (Cf. [12, p. 58]) *be introduced into the quotient space $Q = H/{}_bH$, then ϕ is a continuous mapping.*

Proof. ${}_bH$ is certainly closed, since the topology of H is admissible. Then the natural topology in $Q = H/{}_bH$ is as follows: Let U be an arbitrary open set in H and let ${}_bHU$ denote the set of all cosets ${}_bHg$ where $g \in U$. Then these sets ${}_bHU$ are taken as a basis in Q. Now let an element ${}_bHg$ of Q be given, and a neighborhood W of $\phi({}_bHg) = g(b)$ in A. Now (b, W) is a k-neighborhood of $g \in H$, and thus an open set in H (Cf. Theorem 2). Clearly, if $h \in (b, W)$, then $\phi({}_bHh) = h(b) \in W$. Thus ϕ is continuous.

LEMMA 6. *In order that ϕ^{-1} be continuous, it is necessary and sufficient that for each neighborhood U of the identity in H, and for each $b \in A$, the set $U(b) \subset A$ (that is, the orbit of b under U) have a non-void interior, relative to $H(b)$, which contains b.*

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[18] The condition of separability assumed by van Dantzig and van der Waerden follows from the remaining conditions (Cf. Sierpinski [13, p. 111]).

Proof. Let $x \in H(b) \subset A$ be given, and also a neighborhood ${}_bHUg$ of the coset ${}_bHg \in Q$ be given. Then $\phi({}_bHUg) = U(x)$ and by hypothesis there is an open set W in A such that $x \in W \cap H(b) \subset U(x)$. Obviously $\phi^{-1}(W) \subset {}_bHUg$.

Conversely, suppose that ϕ^{-1} is continuous. Then $\phi({}_bHUg) = U(g(b))$ is an open set (relative to $H(b)$).

Combining these, we infer

THEOREM 8. *If H is an admissibly topologized group of homeomorphisms of a topological space A, and if for every neighborhood U of the identity in H, we have $U(b)$ open relative to $H(b)$, for b in A, then the quotient space $H/{}_bH$ is homeomorphic to the orbit $H(b)$.*

COROLLARY. *If H is a locally compact, admissibly topologized group of homeomorphisms of a topological space A, and if H has a denumerable basis while the orbit $H(b)$ is not the sum of countably many sets which are nowhere dense on $H(b)$, then the quotient space $H/{}_bH$ is homeomorphic to the orbit $H(b)$.*[19]

Proof. Let U be any neighborhood of the identity in H, and let V be a neighborhood of the identity whose closure is compact and contained in U. Now H can be covered by a countable set $V_1, V_2, V_3, \cdots$ of neighborhoods each a translation of V, since it has a countable basis, and so the orbit $H(b)$ is covered by the sets $\bar{V}_1(b), \bar{V}_2(b), \cdots$, which are all homeomorphic to $\bar{V}(b)$, which is closed, and hence nowhere dense if it has no interior (all relative to the orbit $H(b)$). Now select V so that $V^{-1}V \subset U$ (Cf. [12, p. 54]). Then there is an open set W, and a point x in $H(b)$ such that $x \in W \cap H(b) \subset V(b)$. Suppose $x = g(b)$, where $g \in V$. Then $b \in g^{-1}(W \cap H(b)) \subset g^{-1}V(b) \subset U(b)$. Thus the hypothesis of Lemma 6 is fulfilled.

We shall now give an example of a compact metric space A with a locally compact group of homeomorphisms H for which the hypothesis and conclusion of Theorem 8 fail to be fulfilled.

We use cylindrical coordinates (r, θ, z) to describe A in Cartesian 3-space. Let S_n be the closed segment connecting $p_n = (1, n, 0)$ with $q_n = (0, 0, 1 - 2^{-n})$ when $n \geq 1$, and connecting $p_n = (1, n, 0)$ with $q_n = (0, 0, 2^n - 1)$ when $n \leq -1$, and let S_0 connect $p_0 = (1, 0, 0)$ with the origin $q_0 = (0, 0, 0)$. It is essential that θ be measured in radians so that the points $(1, n, 0)$ shall be all distinct and dense on the circle $r = 1$ in the (x, y)-plane.

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[19] This covers the important case in which H is a transitive Lie group of transformations. The proof is an obvious modification of [12, p. 65] (Theorem 13).

Then A is to be the union of all the sets S_n together with the segment connecting $(0, 0, -1)$ with $(0, 0, 1)$ on the z-axis, and the double conical mantle enclosing all the segments whose equation can be written as $|z| + r = 1$, r being always ≥ 0.

It is readily seen that A is closed, and hence compact. But now observe that a homeomorphism h which leaves the end p_n of a segment S_n fixed must leave the other end q_n fixed also. For each n, a homeomorphism h_n can be constructed which rotates the mantle of A through n radians, and sends each S_m into S_{m+n}, p_m into p_{m+n}, q_m into q_{m+n} and acts linearly on the segment connecting q_m with q_{m+1}. Then $H = \{h_n\}$ is a group homeomorphic to the group of integers. Only the identity leaves p_0 fixed, and yet the orbit of p_0, viz. $(\cdots, p_{-1}, p_0, p_1, p_2, \cdots)$ is dense on itself (hence, by Theorem 8, Corollary, must be the sum of a countable number of nowhere dense subsets of itself). Nor can this situation be patched up by choosing a different topology for H, for the orbit of q_0 is homeomorphic to the set of integers.[20]

10. The topology of Lie groups. The question arises: suppose we take a Lie group of transformations, G, and, ignoring the topology of G inherited from its parameter space, introduce the g-topology into G, then how does this topology compare with the original? We shall first introduce the concept of a Lie group of transformations, limiting ourselves to the analytic case (Pontrjagin, [12, p. 287]): An r-dimensional Lie group G is a topological group for which some neighborhood of the identity can be homeomorphically mapped on a neighborhood of Cartesian r-space in such a way that the group operations in G are analytic when expressed in terms of the coordinates $(a_1, \cdots, a_r)$, called parameters, thus introduced into this neighborhood of the identity. G is a Lie group of transformations if each $g \in G$ determines a bicontinuous transformation T of an analytic manifold A, such that if coordinates $(x_1, \cdots, x_n)$ are chosen locally in A, then this transformation T can be expressed by $\bar{x}_i = f_i(x_1, \cdots, x_n; a_1, \cdots, a_n)$ where f_i is continuous in all its arguments simultaneously, and analytic in each.[21] One also demands that T leave each point of A fixed if and only if it corresponds to the identity in G.

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[20] Even under the group of all homeomorphisms of A, the orbits of p_0 and q_0 are not homeomorphic, being the same as under H. And yet the subgroup leaving p_0 invariant coincides with that leaving q_0 fixed. Thus we have a counterexample to a theorem of van Dantzig [5, p. 610], which says, (using the terminology of our Lemma 5) that if A is locally compact and H contains all homeomorphisms of A, then ϕ is a homeomorphism.

[21] One may define all this with functions s times continuously differentiable, rather than analytic. Theorem 9 would require only $s \geq 2$.

THEOREM 9. *Suppose that G is a Lie group of transformations. Let the topology in G be replaced by the g-topology, giving a topological group H of transformations. Then, if G is a compact Lie group, H and G are topologically isomorphic, and, in general, when G is locally compact, the maximal connected subsets of the open sets of H form a basis for G.*

Proof. In the first place, the topology of G is always admissible, and therefore weaker than the k- or g-topologies, which are equivalent here, of course. Therefore, if G is compact, H and G must be homeomorphic (see [12, p. 65]).

For the general case where G is locally compact, we suppose that a neighborhood U of the identity in G be given, such that U^- is compact.

Since, by definition, no element on $U^- \cap U'$ leaves all points of A fixed, and since $U^- \cap U'$ is compact, we must be able to find a finite set of points $p_1, \cdots, p_m$ in A together with open sets $V_1, \cdots, V_m$ where $p_k \in V_k$, $k = 1, \cdots, m$, such that if g lies on $U^- \cap U'$ then the corresponding T drives some p_k out of its V_k, for the following reason: the subgroup ${}_pG$ of G leaving $p \in A$ fixed is closed; therefore some finite set of such subgroups must have a void intersection on the set $U^- \cap U'$. This gives the points $p_1, \cdots, p_m$. To get the $V_1, \cdots, V_m$, one observes that the sum of the distances the p_k are moved by g on the compact set $U^- \cap U'$ is bounded away from zero by a positive number d, and that spheres of radius d/m about the p_k will serve.

It is not hard to see that if U is sufficiently small then this means that if the matrix M is composed by placing m blocks $B_1, B_2, \cdots, B_m$ together

$$M = \| B_1 B_2 \cdots B_m \|$$

where

$$B_k = \begin{Vmatrix} \dfrac{\partial f_1(p_k; a)}{\partial a_1} & \cdots & \dfrac{\partial f_n(p_k; a)}{\partial a_1} \\ \cdot & & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & & \cdot \\ \dfrac{\partial f_1(p_k; a)}{\partial a_r} & \cdots & \dfrac{\partial f_n(p_k; a)}{\partial a_r} \end{Vmatrix}$$

then M will have a minor or rank r which does not vanish on U.

This means that we can solve for the $(a_1, \cdots, a_r)$ in terms of some set of r coordinates of the transforms $\bar{p}_k$ of the p_k, this inversion:

$$a_i = \phi_i(\bar{p}_1, \cdots, \bar{p}_m)$$

being continuous and valid for $\bar{p}_k \in V_k$, where perhaps the V_k must be diminished in radius.

Now suppose that L is a connected set in H lying in the k-neighborhood $(p_1, \cdots, p_m; V_1, \cdots, V_m)$, and containing the identity in H. Then $L(p_k) \subset V_k$, and hence, by the inversion $a_i = \phi_i(\bar{p}_1, \cdots, \bar{p}_m)$, determines a connected set S in G, containing the identity. If S met the set $U^- \cap U'$, some p_k would be driven out of its V_k, whence $S \subset U$. This means that the connected component of $(p_1, \cdots, p_m; V_1, \cdots, V_m)$ lies in U, thus proving the theorem.

REMARK. The part of this theorem relating to compact Lie groups of transformations can be extended to any compact, effective (i. e., only the identity of the group leaves all points of A fixed) group of transformations, provided its topology is admissible (this is sometimes included in the concept "transformation group" (Cf. [9, p. 365])), since the proof for that part involves only the admissibility and compactness of the topology.

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ON A CLASS OF PARTIAL DIFFERENTIAL EQUATIONS OF EVEN ORDER.*

By JÓAQUIN B. DIAZ.

1. Introduction.[1] In this paper, we consider linear partial differential equations of the form

$$L^N u = 0, \tag{1.1}$$

where $u = u(x, y)$ and L is a certain linear partial differential operator of the second order. The most important special case is that of

$$Lu = \Delta u + \phi(x)u_x + \psi(y)u_y, \tag{1.2}$$

where $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$; then (1.1) becomes

$$L^N u = \Delta^N u + \text{lower order terms} = 0.$$

Our main purpose is to construct a sequence of particular solutions of (1.1) having the following properties: (a) In the neighborhood of each point at which a solution of the equation $L^N u = 0$ is analytic, this solution can be expanded in a uniformly and absolutely convergent series of linear combinations of these particular solutions; (b) Each particular solution is defined and analytic in the whole plane and can be obtained by quadratures from the coefficients of L. Furthermore, all of these particular solutions are linear combinations of certain functions independent of N.

Our method is based upon the solution of partial differential equations with constant coefficients by the method of hypercomplex variables. [See Scheffers (3) [2], and Ketchum (2); Ward (5) gives an exhaustive bibliography on analytic functions of linear algebras.] This method involves replacing the given partial differential equation of higher order by a system of partial differential equations of the first order. Sobrero (4) has treated in detail the equation $\Delta^2 u = 0$ from this standpoint. For the equation

$$\Delta^N u = 0 \tag{1.3}$$

the procedure is as follows. Formally, we determine the hypercomplex

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[1] Presented to the American Mathematical Society, September 17, 1945. The problem was suggested to the author by Dr. Lipman Bers.

[2] Numbers in parenthesis refer to papers in the bibliography.

unit j_N so that every function $u = f(x, y) = f(x + j_N y)$ shall be a solution of (1.3). By direct substitution, setting $f^{(2N)} = \dfrac{d^{2N} f}{d^{2N}(x + j_N y)}$, we obtain

$$\Delta^N f = \sum_{l=0}^{N} \binom{N}{l} \frac{\partial^{2N} f}{\partial x^{2(N-l)} \partial y^{2l}} = f^{(2N)} \sum_{l=0}^{N} \binom{N}{l} j_N{}^{2l}.$$

Thus, the necessary and sufficient condition that every function $f(x, y) = f(x + j_N y)$ be a solution of (1.3) is that

$$(1 + j_N{}^2)^N = 0. \tag{1.4}$$

From (1.4), it is seen that any power of j_N greater than the $2N$-th may be expressed as a linear combination of $1, j_N, \cdots, j_N{}^{2N-1}$. The hypercomplex valued function $f(x, y) = \sum_{i=0}^{2N-1} a_i(x, y) j_N{}^i$ is said to be analytic at (x_0, y_0) provided there exists a neighborhood of (x_0, y_0) such that if (x_1, y_1) is any point of this neighborhood, the difference quotient

$$\frac{f(x, y) - f(x_1, y_1)}{(x - x_1) + j_N(y - y_1)}$$

approaches a limit independent of the mode of approach of the point (x, y) to the point (x_1, y_1). Letting (x, y) approach (x_1, y_1) in the directions parallel to the x and y axes, and equating the results, we obtain

$$\partial f/\partial x = (1/j_N)(\partial f/\partial y);$$

that is

$$\sum_{i=0}^{2N-1} a_{i,x} j_N{}^i = (1/j_N) \sum_{i=0}^{2N-1} a_{i,y} j_N{}^i,$$

where $a_{i,x} = \partial a_i/\partial x$; etc. Multiplying through by j_N in the last equation and employing (1.4) in the form

$$j_N{}^{2N} = -\sum_{i=0}^{N-1} \binom{N}{i} j_N{}^{2i},$$

we obtain

$$\sum_{k=0}^{N-1} [a_{2k-1,x} - a_{2N-1,x} \binom{N}{k}] j_N{}^{2k} + \sum_{k=0}^{N-1} a_{2k,x} j_N{}^{2k+1}$$
$$= \sum_{k=0}^{N-1} a_{2k,y} j_N{}^{2k} + \sum_{k=0}^{N-1} a_{2k+1,y} j_N{}^{2k+1}.$$

To simplify the writing, we set $a_{-1} \equiv 0$. This device of introducing zero terms will be used throughout, without explicit mention. Equating coefficients of like powers of j_N, we see that

$$a_{i,y} = a_{2k-1,x} - a_{2N-1,x}\binom{N}{k}, \tag{1.5}$$

$(k = 0, \cdots, N-1)$. The system (1.5) bears the same relation to (1.3) as the Cauchy-Riemann equations bear to Laplace's equation. To complete the analogy, it remains to show that each of the $2N$ functions a_i satisfies (1.3), and that given any solution u of (1.3), there is a solution a_i, $i = 0, \cdots, 2N-1$, of (1.5) such that $a_0 = u$.

Returning now to (1.2), we rewrite L in the form

$$Lu = (\tau/\sigma)[(\sigma u_x/\tau)_x + (\sigma u_y/\tau)_y], \tag{1.6}$$

where

$$\sigma = \exp\left(\int \phi dx\right), \qquad \tau = \exp\left(-\int \psi dy\right).$$

Actually, we shall consider the more general case

$$Lu = (\tau_1/\sigma_2)[(\sigma_1 u_x/\tau_1)_x + (\sigma_2 u_y/\tau_2)_y]. \tag{1.7}$$

If σ_i and τ_i do not change sign, the σ_i's have the same sign, and the τ_i's have the same sign, then (1.7) can be transformed into (1.6) by the change of variables

$$\xi = \int (\sigma_2/\sigma_1)^{1/2} dx, \qquad \eta = \int (\tau_2/\tau_1)^{1/2} dy.$$

We prefer to treat the general case because no additional difficulties arise and the formal part of our considerations remains valid even when the σ's and τ's change sign. In particular, this is true when L is of hyperbolic or of mixed type.

The system of first order equations connected with (1.1) is

$$\begin{aligned} \sigma_1(x) a_{2k,x} &= \tau_1(y) a_{2k+1,y} \\ \sigma_2(x) a_{2k,y} &= \tau_2(y)\left[a_{2k-1,x} - a_{2N-1,x}\binom{N}{k}\right], \end{aligned} \tag{1.8}$$

$(k = 0, \cdots, N-1)$, which arises in a natural way from (1.5).

Bers and Gelbart (1) have studied the system of equations [3]

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[3] A system of equations of the form (1.9), with $\sigma_1 = \sigma_2 = \tau_1 = 1$, and which appears in the theory of compressible fluids, has been discussed independently by Professor S. Bergman, who introduced an operator analogous to the "formal powers" employed by Bers and Gelbart [1]. See S. Bergman, "The hodograph Method in the Theory of Compressible Fluids" (Supplement to R. V. Mises and K. O. Friedrichs,

$$\begin{aligned} \sigma_1(x)u_x &= \tau_1(y)v_y, \\ \sigma_2(x)u_y &= -\tau_2(y)v_x, \end{aligned} \tag{1.9}$$

which is obtained from the Cauchy-Riemann equations just as (1.8) is obtained from (1.5). By eliminating v from (1.9), we obtain the equation $Lu = 0$.

To simplify the writing, we use the following real functions

$$\begin{aligned} X(x) &= \int_0^x (1/\sigma_1(\xi))\,d\xi, & Y(y) &= \int_0^y (1/\tau_1(\eta))\,d\eta, \\ X^*(x) &= \int_0^x \sigma_2(\xi)\,d\xi, & Y^*(y) &= \int_0^y \tau_2(\eta)\,d\eta, \end{aligned} \tag{1.10}$$

and rewrite (1.8) in the form

$$\begin{aligned} a_{2k,X} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= a_{2k-1,X^*} - a_{2N-1,X^*}\binom{N}{k}, \end{aligned} \tag{1.11}$$

$(k = 0, \cdots, N-1)$, where $a_{2k,X} = \partial a_{2k}/\partial X$, etc. We shall refer to (1.8), or (1.11), as the system $E(\Sigma, N)$, where Σ is the matrix of the coefficients of (1.8)

$$\Sigma = \begin{Vmatrix} \sigma_1 & \tau_1 \\ \sigma_2 & \tau_2 \end{Vmatrix}. \tag{1.12}$$

We remark that if $u = u(x, y)$ has continuous second derivatives, then $\partial^2 u/\partial X \partial Y = \partial^2 u/\partial Y \partial X$, but that in general $\partial^2 u/\partial X \partial X^* \neq \partial^2 u/\partial X^* \partial X$, etc. With the aid of (1.10), the operator L of (1.7) may be rewritten in the form

$$Lu = u_{XX^*} + u_{Y^*Y}.$$

Together with (1.8), we shall be led to consider the system

$$\begin{aligned} (1/\sigma_2(x))\,a_{2k,x} &= \tau_1(y)\,a_{2k+1,y}, \\ (1/\sigma_1(x))\,a_{2k,y} &= \tau_2(y)\left[a_{2k-1,x} - a_{2N-1,x}\binom{N}{k}\right], \end{aligned} \tag{1.13}$$

$(k = 0, \cdots, N-1)$, which may be rewritten in the form

$$\begin{aligned} a_{2k,X^*} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= a_{2k-1,X} - a_{2N-1,X}\binom{N}{k}. \end{aligned} \tag{1.14}$$

Fluid Dynamics), mimeographed lecture notes, Brown University, 1942, especially pages 23-24; and "A Formula for the Stream Function of Certain Flows," *Proceedings of the National Academy of Sciences*, vol. 29 (1943), pp. 276-281, by the same author.

$(k = 0, \cdots, N-1)$. We shall refer to (1.13), or (1.14), as the system $E(\Sigma', N)$, where Σ' is the matrix of the coefficients of (1.13)

$$\Sigma' = \left\| \begin{matrix} (1/\sigma_2) & \tau_1 \\ (1/\sigma_1) & \tau_2 \end{matrix} \right\|. \tag{1.15}$$

It will be assumed throughout that the functions $\sigma_1(x)$, $\sigma_2(x)$, $\tau_1(y)$, and $\tau_2(y)$ are *positive analytic functions defined for all values of their respective real variables,* but it is clear that some of the results hold under weaker hypotheses. All the hypercomplex valued functions we shall deal with are defined on the (x, y) plane. Accordingly, we shall write $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ for a typical function, where $z = (x, y) = x + iy$ is a variable point of the (x, y)-plane and $a_i = a_i(x, y)$, and speak of the domain in the plane on which f is defined. By $| z |$ is understood the distance of the point z from the origin and by a domain is understood an open connected set.

In order to construct a function theory for (1.11), it is first necessary to study the algebra generated by the unit j_N of (1.4). This is done in Section **2**. The real algebra of order $2N$ generated by j_N is shown to be isomorphic to a certain complex algebra of order N; and a criterion for a number of the algebra to possess an inverse is given. In Section **3** (Σ, N)-monogenic functions are introduced. These are functions $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ such that the a_i satisfy the system $E(\Sigma, N)$, and they take the place of analytic functions. For these functions, the processes of (Σ, N)-differentiation and (Σ, N)-integration are defined. These two processes transform (Σ, N)-monogenic functions into (Σ', N)-monogenic functions [see (1.14) and (1.15)]. The process of "contraction," which transforms (Σ, N)-monogenic functions into $(\Sigma, N-M)$-monogenic functions, is treated in Section **4**. The relations between (Σ, N)-, (Σ', N)-, $(\Sigma_,, N)$-, and $(\Sigma_,', N)$-monogenic functions are discussed in Section **6**, where the matrices of coefficients $\Sigma_,$ and $\Sigma'_,$ are defined. In Section **5** it is proved that if $f(z)$ is (Σ, N)-monogenic, then the real functions a_{2k} satisfy the equation $L^N u = 0$. The process of (Σ, N)-integration enables us to construct special (Σ, N)-monogenic functions, called "formal powers," which bear the same relation to $E(\Sigma, N)$ as the powers of $x + iy$ bear to the Cauchy-Riemann equations. By means of these "formal powers," it is possible (Section **8**) to characterize completely all (Σ, N)-monogenic functions for which one of the functions a_i vanishes identically ("null" functions). It is then proved (Section **9**) that every analytic function satisfying $L^N u = 0$ is an "even component" of a (Σ, N)-

monogenic function determined to within a "null function." The main result concerning (Σ, N)-monogenic functions is the expansion theorem proved in Section **10**, which states that a (Σ, N)-monogenic function may be expanded uniquely in a formal power series in the neighborhood of any point at which the function is (Σ, N)-monogenic. Using this expansion theorem, we construct the sequence of particular solutions of $L^N u = 0$ mentioned at the outset.

2. The algebras A_{2N}.

A. For $N = 1, 2, 3, \cdots$ let A_{2N} denote the algebra of order $2N$ over the real field, whose elements are the polynomials

$$\sum_{i=0}^{2N-1} a_i j_N{}^i, \tag{2.1}$$

where the a_i are real numbers and the unit j_N satisfies the equation

$$(1 + j_N{}^2)^N = 0. \tag{2.2}$$

Equality is defined by postulating that

$$\sum_{i=0}^{2N-1} a_i j_N{}^i = \sum_{i=0}^{2N-1} b_i j_N{}^i,$$

if, and only if, $a_i = b_i$ for $n = 0, \cdots, 2N - 1$.

Addition is defined by the identity

$$\sum_{i=0}^{2N-1} a_i j_N{}^i + \sum_{i=0}^{2N-1} b_i j_N{}^i = \sum_{i=0}^{2N-1} (a_i + b_i) j_N{}^i.$$

Multiplication is defined by the identity

$$\left(\sum_{i=0}^{2N-1} a_i j_N{}^i\right)\left(\sum_{i=0}^{2N-1} b_i j_N{}^i\right) = \sum_{i=0}^{4N-2} \left(\sum_{l+k=i} a_l b_k\right) j_N{}^i = \sum_{i=0}^{2N-1} c_i j_N{}^i,$$

where the c_i's are obtained by employing the relation [see (2.2)]

$$j_N{}^{2N} = -\sum_{k=0}^{N-1} \binom{N}{k} j_N{}^{2k}. \tag{2.3}$$

For each N, the system possesses the identity element 1. If a number of A_{2N} has an inverse, then the inverse is unique.

Let x be an indeterminate over the real field. Then, for any $N = 1, 2, 3, \cdots$

$$\sum_{k=0}^{N-1} a_{2k} x^{2k} + \sum_{k=0}^{N-1} a_{2k+1} x^{2k+1} \equiv \sum_{l=0}^{N-1} (\alpha_l + x\beta_l)(1 + x^2)^l, \tag{2.4}$$

where the real numbers α_l and β_l are uniquely determined by

$$\alpha_l = \sum_{k=l}^{N-1} (-1)^{k-l} \binom{k}{l} a_{2k},$$
$$\beta_l = \sum_{k=l}^{N-1} (-1)^{k-l} \binom{k}{l} a_{2k+1}, \tag{2.5}$$

$(l = 0, 1, \cdots, N-1)$. Thus, we have

THEOREM 2.1. *Every number* $\sum_{i=0}^{2N-1} a_i j_N{}^i$ *of* A_{2N} *may be expressed in a unique way in the form* $\sum_{l=0}^{N-1} (\alpha_l + j_N \beta_l)(1 + j_N{}^2)^l$.

B. In what follows, we shall make use of the fact that $j_N{}^2$ has an inverse. In order to obtain this inverse, we need the binomial identity

$$\sum_{k=q-1}^{N-1} \binom{k}{q-1} = \binom{N}{q}, \tag{2.6}$$

where $N = 1, 2, \cdots$ and $1 \leq q \leq N-1$. Using (2.6), we verify that

$$1/j_N{}^2 = -\sum_{k=0}^{N-1} (1 + j_N{}^2)^k. \tag{2.7}$$

C. Let the sequence of positive real numbers p_i be defined by the equations

$$\sum_{i+l=k} p_i p_l = 1, \qquad p_0 = 1, \tag{2.8}$$

$k = 0, 1, 2, \cdots$. It follows that $p_i = (-1)^i \binom{-\frac{1}{2}}{i}$, since $(\sum_{i=0}^{\infty} p_i x^i)^2 = 1/1 - x$ for $|x| < 1$, but the exact values of the p_i will not be needed. We define

$$i_N = j_N \sum_{i=0}^{N-1} p_i (1 + j_N{}^2)^i \tag{2.9}$$

for $N = 1, 2, \cdots$. Employing (2.7) and (2.9), we have

THEOREM 2.2. *The numbers* i_N *and* $-i_N$ *of the algebra* A_{2N} *satisfy the equation* $x^2 = -1$.

THEOREM 2.3. *Every number* $\sum_{k=0}^{N-1} (\alpha_k + j_N \beta_k)(1 + j_N{}^2)^k$ *of* A_{2N} *may be expresed in a unique way in the form* $\sum_{k=0}^{N-1} (\gamma_k + i_N \delta_k)(1 + j_N{}^2)^k$, *where* i_N *is given by* (2.9); *the real numbers* γ_k *and* δ_k *are given by*

$$\gamma_k = \alpha_k,$$

$$\sum_{i+l=k} \delta_i p_l = \beta_k, \tag{2.10}$$

$(k = 0, 1, \cdots, N-1)$, *and the* p_l *are given by* (2.8).

This may be seen at once by direct computation. The last result may be stated in another way. For $N = 1, 2, 3, \cdots$ let B_N denote the algebra of order N over the complex field whose elements are the polynomials

$$\sum_{i=0}^{N-1} b_i \omega_N{}^i,$$

where the b_i are complex numbers and the unit ω_N satisfies the equation

$$(\omega_N)^N = 0.$$

Equality is defined by postulating that

$$\sum_{i=0}^{N-1} b_i \omega_N{}^i = \sum_{i=0}^{N-1} c_i \omega_N{}^i,$$

if, and only if, $b_i = c_i$ for $i = 0, 1, \cdots, N-1$. Addition and multiplication are defined by the identities

$$\sum_{i=0}^{N-1} b_i \omega_N{}^i + \sum_{i=0}^{N-1} c_i \omega_N{}^i = \sum_{i=0}^{N-1} (b_i + c_i) \omega_N{}^i,$$

$$\Big(\sum_{i=0}^{N-1} b_i \omega_N{}^i\Big)\Big(\sum_{i=0}^{N-1} c_i \omega_N{}^i\Big) = \sum_{i=0}^{N-1} \Big(\sum_{l+k=i} b_l c_k\Big) \omega_N{}^i. \tag{2.11}$$

Theorem 2.3 states that for $N = 1, 2, 3, \cdots$ *the algebra* A_{2N} *of order* $2N$ *over the real field is isomorphic to the algebra* B_N *of order* N *over the complex field.* In this isomorphism, the number $\sum_{k=0}^{N-1} (\alpha_k + j_N \beta_k)(1 + j_N{}^2)^k$ of A_{2N} corresponds to the number $\sum_{k=0}^{N-1} (\gamma_k + i\delta_k) \omega_N{}^k$ of B_N, where the γ_k and δ_k are given by (2.10) and $i = j_1 = i_1$ is the ordinary complex unit.

D. Using the same notation as above, let

$$\sum_{k=0}^{N-1} a_{2k} j_N{}^{2k} + \sum_{k=0}^{N-1} a_{2k+1} j_N{}^{2k+1} = \sum_{k=0}^{N-1} (\alpha_k + j_N \beta_k)\ (1 + j_N{}^2)^k,$$

be a number of A_{2N}. Recalling (2.5) and (2.10), and keeping in mind that $p_0 = 1$ from (2.8), we have

$$\gamma_0 = \alpha_0 = \sum_{k=0}^{N-1} (-1)^k a_{2k},$$

$$\delta_0 = \beta_0 = \sum_{k=0}^{N-1} (-1)^k a_{2k+1}. \tag{2.12}$$

THEOREM 2.4. *The number* $\sum_{i=0}^{2N-1} a_i j_N{}^i = \sum_{k=0}^{N-1} (\gamma_k + i_N \delta_k)(1 + j_N{}^2)^k$ *of* A_{2N} *has an inverse if, and only if, the ordinary complex number* $\gamma_0 + i\delta_0$ *does not vanish.*

Proof. In view of the isomorphism between A_{2N} and B_N, it suffices to show that the asserted condition is necessary and sufficient for the number $\sum_{i=0}^{N-1} b_i \omega_N{}^i = \sum_{k=0}^{N-1} (\gamma_k + i\delta_k)\omega_N{}^k$ of B_N to have an inverse.

Suppose that $b_0 = \gamma_0 + i\delta_0 = 1/c_0$ and define the complex numbers c_i by

$$b_0 c_0 = 1,$$

$$\sum_{l+k=i} b_l c_k = 0, \tag{2.13}$$

$(i = 1, 2, \cdots, N-1)$. Then, by (2.11), $\sum_{i=0}^{N-1} c_i \omega_N{}^i$ is the desired inverse. Conversely, if $\sum_{i=0}^{N-1} b_i \omega_N{}^i$ has the inverse $\sum_{k=0}^{N-1} c_i \omega_N{}^i$ then, from (2.11), equations (2.13) hold and c_0 is the inverse of b_0.

Taking (2.12) into account, it follows that the number $\sum_{i=0}^{2N-1} a_i j_N{}^i$ of A_{2N} has an inverse if, and only if,

$$[\sum_{k=0}^{N-1} (-1)^k a_{2k}]^2 + [\sum_{k=0}^{N-1} (-1)^k a_{2k+1}]^2 > 0. \tag{2.14}$$

THEOREM 2.5. *If a number* $\sum_{i=0}^{2N-1} a_i j_N{}^i$ *of* A_{2N} *which is the sum of even (odd) powers of* j_N *has an inverse, then its inverse is also the sum of even (odd) powers of* j_N.

Proof. Let

$$\sum_{k=0}^{N-1} a_{2k} j_N{}^{2k} + \sum_{k=0}^{N-1} a_{2k+1} j_N{}^{2k+1} = \sum_{k=0}^{N-1} (\alpha_k + j_N \beta_k)(1 + j_N{}^2)^k$$

$$= \sum_{k=0}^{N-1} (\gamma_k + i_N \delta_k)(1 + j_N{}^2)^k, \tag{2.15}$$

in A_{2N} and let

$$\sum_{i=0}^{N-1} b_i \omega_N^i = \sum_{k=0}^{N-1} (\gamma_k + i\delta_k)\omega_N^k, \tag{2.16}$$

be the image of $\sum_{i=0}^{2N-1} a_i j_N^i$ in B_N.

We note that all the $a_{2k} = 0$ in (2.15) if, and only if, all the b_i are purely imaginary in (2.16). Analogously, all the $a_{2k+1} = 0$ in (2.15) if, and only if, all the b_i are real in (2.16).

For, if all $a_{2k} = 0$ in (2.15), then all $\alpha_k = 0$ from (2.5), and all $\gamma_k = 0$ from (2.10). Hence, all the b_i of (2.16) are purely imaginary. Conversely, if all the b_i of (2.16) are purely imaginary, then all $\gamma_k = 0$ in (2.16), all $\alpha_k = 0$ from (2.10), and all $a_{2k} = 0$ from (2.5). Similarly for the a_{2k+1}.

Let the inverse of $\sum_{i=0}^{N-1} b_i \omega_N^i$ be denoted by $\sum_{i=0}^{N-1} c_i \omega_N^i$. To complete the proof, it is merely necessary to notice that in view of (2.11) and (2.13), if all the b_i are real (purely imaginary), then all the c_i are also real (purely imaginary), and to apply the above reasoning to $\sum_{i=0}^{N-1} c_i \omega_N^i$.

E. THEOREM 2.6. *If* $f = \sum_{i=0}^{2N-1} a_i j_N^i$ *and* $g = j_N f = \sum_{i=0}^{2N-1} b_i j_N^i$, *then*

$$\begin{aligned} b_{2k} &= a_{2k-1} - \binom{N}{k} a_{2N-1}, \\ b_{2k+1} &= a_{2k}, \end{aligned} \tag{2.17}$$

$(k = 0, \cdots, N-1)$, *and*

$$\begin{aligned} a_{2k} &= b_{2k+1}, \\ a_{2k+1} &= b_{2k+2} - \binom{N}{k+1} b_0, \end{aligned} \tag{2.18}$$

$(k = 0, \cdots, N-1)$.

(2.17) follows by direct computation, and (2.18) is obtained from it by observing that for $k = 0$ the first equation of (2.17) gives

$$b_0 = -a_{2N-1}.$$

We remark that if $g = j_N f$, then $f = g/j_N$, since j_N has an inverse, by (2.14)

Let $f = \sum_{i=0}^{2N-1} a_i j_N^i$ be a number of A_{2N}. We shall use the following notation:

$$\begin{aligned} Re[f] &= a_0, \\ Im[f] &= a_1. \end{aligned} \tag{2.19}$$

THEOREM 2.7. *Let* $f = \sum_{i=0}^{2N-1} a_i j_N^i$ *be a number of* A_{2N}, *and define*

$$K_{N,k} = -\sum_{l=k+1}^{N} \binom{N}{l} j_N^{2(l-k)}, \tag{2.20}$$

$$F_{N,k} = K_{N,k} f,$$

$(k = 0, \cdots, N-1)$. *Then*

$$Re[F_{N,k}] = a_{2k}, \qquad Im[F_{N,k}] = a_{2k+1}, \tag{2.21}$$

$(k = 0, \cdots, N-1)$.

Proof. From the definition (2.20), we have at once

$$K_{N,k} = j_N^2 K_{N,k+1} - j_N^2 \binom{N}{k+1}, \tag{2.22}$$

$(k = 0, 1, \cdots, N-2)$. The proof of the theorem now follows by an induction on k. First, $K_{N,k}$, for $k = N-1$, has the desired property (2.21), by (2.20) and (2.18). The induction is completed with the aid of these same two equations plus (2.22).

THEOREM 2.8. *$K_{N,k}$ has an inverse in A_{2N} which is the sum of even powers of j_N.*

Proof. The existence of the inverse follows immediately from (2.14), the definition of $K_{N,k}$, and the identity

$$-\sum_{l=k+1}^{N} \binom{N}{l} (-1)^{l-k} = \binom{N-1}{k},$$

$(k = 0, \cdots, N-1)$. That $1/K_{N,k}$ is a sum of even powers of j_N is seen from theorem 2.5.

F. In later sections, we shall employ the concept of the "absolute value," or norm, of a number of A_{2N}. Let $\alpha = \sum_{i=0}^{2N-1} a_i j_N^i$ be a number of A_{2N}. The absolute value of α, denoted by $\| \alpha \|$, is a non-negative number such that

$$\| \alpha \|^2 = \sum_{i=0}^{2N-1} a_i^2. \tag{2.23}$$

From this definition, we have the "triangle inequality"

$$| \| \alpha \| - \| \beta \| | \leqq \| \alpha + \beta \| \leqq \| \alpha \| + \| \beta \|. \tag{2.24}$$

If c is a real number, then

$$\| c\alpha \| = | c | \, \| \alpha \|. \tag{2.25}$$

There is a useful inequality which gives an upper bound for the absolute

value of the product of two numbers of A_{2N}. There exists a positive constant Γ_N such that [see (2), page 643]

$$\| \alpha\beta \| \leqq \Gamma_N \| \alpha \| \| \beta \|, \tag{2.26}$$

for any α and β of A_{2N}. Therefore

$$\| \alpha^n \| \leqq \Gamma_N{}^{n-1} \| \alpha \|^n. \tag{2.27}$$

G. Any hypercomplex number in the algebra A_{2N} may be written as a polynomial in j_N of degree at most $2N-1$. It is desirable to have a formula giving the powers of j_N greater than the $2N-1$-st as sums of powers of j_N ranging from 0 to $2N-1$. We have

$$\begin{aligned} j_N{}^{2\nu} &= (-1)^{N+\nu+1} \sum_{k=0}^{N-1} \binom{\nu}{k} \binom{\nu-1-k}{N-1-k} j_N{}^{2k}, \\ & \qquad (\nu = N, N+1, \cdots) \\ j_N{}^{2\nu+1} &= (-1)^{N+\nu+1} \sum_{k=0}^{N-1} \binom{\nu}{k} \binom{\nu-1-k}{N-1-k} j_N{}^{2k+1}. \end{aligned} \tag{2.28}$$

Only the first formula of (2.28) need be proved, since the second is a consequence of the first.

The proof is by induction, using the binomial identity

$$\binom{\nu}{k-1}\binom{\nu-k}{N-k} + \binom{\nu+1}{k}\binom{\nu-k}{N-k-1} = \binom{\nu}{N-1}\binom{N}{k} \tag{2.29}$$

H. Expanding $(x + j_N y)^p$ by means of the binomial formula and replacing the powers of j_N above the $2N-1$-st by powers of j_N from the 0-th to the $2N-1$-st (using 2.28), we obtain

$$\begin{aligned} (x+j_N y)^p = &\sum_{k=0}^{N-1} \left\{ \binom{p}{2k} y^{2k} x^{p-2k} + \sum_{\nu=N}^{[p/2]} \left[(-1)^{N+\nu+1} \binom{p}{2\nu} \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} y^{2\nu} x^{p-2\nu} \right] \right\} \\ &+ \sum_{k=0}^{N-1} \left\{ \binom{p}{2k+1} y^{2k+1} x^{p-2k-1} + \sum_{\nu=N}^{[(p-1)/2]} \left[(-1)^{N+\nu+1} \binom{p}{2\nu+1} \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} \right. \right. \\ &\left. \left. y^{2\nu+1} x^{p-2\nu-1} \right] \right\} j_N{}^{2k+1}. \end{aligned} \tag{2.30}$$

For later reference, we shall write down (2.30) for even and odd p

$$\begin{aligned} (x+j_N y)^{2q} = &\sum_{k=0}^{N-1} \left\{ \binom{2q}{2k} y^{2k} x^{2q-2k} + \sum_{\nu=N}^{q} \left[(-1)^{N+\nu+1} \binom{2q}{2\nu} \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} \right. \right. \\ &\left. \left. y^{2\nu} x^{2q-2\nu} \right] \right\} j_N{}^{2k} + \sum_{k=0}^{N-1} \left\{ \binom{2q}{2k+1} y^{2k+1} x^{2q-2k-1} + \sum_{\nu=N}^{q-1} \left[(-1)^{N+\nu+1} \right. \right. \\ &\left. \left. \binom{2q}{2\nu+1} \binom{\nu}{k} \binom{\nu-k-1}{N-k-1} y^{2\nu+1} x^{2q-2\nu-1} \right] \right\} j_N{}^{2k+1}, \end{aligned} \tag{2.31}$$

$$(2.32)\quad (x+j_Ny)^{2q+1}=\sum_{k=0}^{N-1}\{\binom{2q+1}{2k}y^{2k}x^{2q+1-2k}+\sum_{\nu=N}^{q}[(-1)^{N+\nu+1}\binom{2q+1}{2\nu}\binom{\nu}{k}\binom{\nu-k-1}{N-k-1}y^{2\nu}x^{2q+1-2\nu}]\}j_N{}^{2k}+\sum_{k=0}^{N-1}\{\binom{2q+1}{2k+1}y^{2k+1}x^{2q-2k}+\sum_{\nu=N}^{q}[(-1)^{N+\nu+1}\binom{2q+1}{2\nu+1}\binom{\nu}{k}\binom{\nu-k-1}{N-k-1}y^{2\nu+1}x^{2q-2\nu}]\}j_N{}^{2k+1}$$

To simplify the writing, we set

$$(2.33)\quad \begin{aligned}(x+j_Ny)^p&\equiv\sum_{i=0}^{2N-1}[\sum_{l=0}^{p}A_{ilp}{}^{(N)}y^lx^{p-l}]j_N{}^i,\\ j_N(x+j_Ny)^p&\equiv\sum_{i=0}^{2N-1}[\sum_{l=0}^{p}B_{ilp}{}^{(N)}y^lx^{p-l}]j_N{}^i,\end{aligned}$$

where the $A_{ilp}{}^{(N)}$ are given explicitly by (2.30) and the $B_{ilp}{}^{(N)}$ may be obtained immediately using Theorem 2.6.

3. (Σ, N)-monogenic functions. Differentiation and integration.

A. For each positive integer N, we consider the system of equations $E(\Sigma, N)$

$$(3.1)\quad \begin{aligned}\sigma_1(x)a_{2k,x}&=\tau_1(y)a_{2k+1,y},\\ \sigma_2(x)a_{2k,y}&=\tau_2(y)[a_{2k-1,x}-a_{2N-1,x}\binom{N}{k}],\end{aligned}$$

$(k=0,\cdots,N-1)$, where $a_i=a_i(x,y)$ for $i=0,\cdots,2N-1$ and $a_{-1}\equiv 0$.

The system (3.1) may be written in a different way. For $k=0$, the second equation of (3.1) yields

$$\sigma_2(x)a_{0,y}=-\tau_2(y)a_{2N-1,x}.$$

With the aid of this last equation, (3.1) becomes

$$(3.2)\quad \begin{aligned}\tau_2(y)a_{2k+1,x}&=\sigma_2(x)[a_{2k+2,y}-a_{0,y}\binom{N}{k+1}],\\ \tau_1(y)a_{2k+1,y}&=\sigma_1(x)a_{2k,x},\end{aligned}$$

$(k=0,\cdots,N-1)$, where $a_{2N}\equiv 0$.

To simplify the writing, we shall use the condensed notation introduced in **1** and rewrite the last two equations in the form

$$(3.3)\quad \begin{aligned}a_{2k,X}&=a_{2k+1,Y},\\ a_{2k,Y^*}&=a_{2k-1,X^*}-a_{2N-1,X^*}\binom{N}{k},\end{aligned}$$

$(k = 0, \cdots, N-1)$, and

$$(3.4) \qquad \begin{aligned} a_{2k+1,X^\bullet} &= a_{2k+2,Y^\bullet} - a_{0,Y^\bullet}\binom{N}{k+1}, \\ a_{2k+1,Y} &= a_{2k,X}, \end{aligned}$$

$(k = 0, \cdots, N-1)$.

By analogy with the theory of functions of a complex variable, we lay down the following definitions:

DEFINITION 3.1. *A hypercomplex valued function* $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$, *where* $a_i = a_i(x, y)$ *for* $i = 0, \cdots, 2N-1$, *is said to be* (Σ, N)*-monogenic at a point* $z = (x, y)$ *if there exists a neighborhood* D *of* z *such that the functions* a_i *belong to class* $C^{(1)}$ *in* D *and satisfy the system* $E(\Sigma, N)$ *in* D.

We shall say that $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic in the domain D if $f(z)$ is (Σ, N)-monogenic at all points of D. With this understanding, we have the

DEFINITION 3.2. *If the function* $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ *is* (Σ, N)*-monogenic in a domain* D *and all the* a_i *belong to class* $C^{(2)}$ *in* D, *then the function* $f^{[1]}(z) = \sum_{i=0}^{2N-1} a_i{}^{[1]} j_N{}^i$, *where*

$$(3.5) \qquad \begin{aligned} a_{2k}{}^{[1]} &= a_{2k,X} = a_{2k+1,Y}, \\ a_{2k+1}{}^{[1]} &= a_{2k+1,X^\bullet} = a_{2k+2,Y^\bullet} - a_{0,Y^\bullet}\binom{N}{k+1}, \end{aligned}$$

$(k = 0, \cdots, N-1)$, *is said to be the* (Σ, N)*-derivative of* $f(z)$ *and is denoted by* $d_{(\Sigma,N)} f / d_{(\Sigma,N)} z$ *or more simply by* $d_\Sigma f / d_\Sigma z$ *when no confusion as to the value of* N *might arise.*

From (3.5) and Theorem 2.6, we may write

$$\begin{aligned} f^{[1]}(z) &= \sum_{k=0}^{N-1} a_{2k,X} j_N{}^{2k} + \sum_{k=0}^{N-1} a_{2k+1,X^\bullet} j_N{}^{2k+1} \\ &= \sum_{k=0}^{N-1} a_{2k+1,Y} j_N{}^{2k} + \sum_{k=0}^{N-1} \left[a_{2k+2,Y^\bullet} - a_{0,Y^\bullet}\binom{N}{k+1}\right] j_N{}^{2k+1} \\ &= (1/j_N)\left[\sum_{k=0}^{N-1} a_{2k,Y^\bullet} j_N{}^{2k} + \sum_{k=0}^{N-1} a_{2k+1,Y} j_N{}^{2k+1}\right]. \end{aligned}$$

The following remarks are immediate consequences of the last two definitions: Any hypercomplex constant $\sum_{i=0}^{2N-1} b_i j_N{}^i$ is (Σ, N)-monogenic, and

conversely any (Σ, N)-monogenic function whose (Σ, N)-derivative vanishes identically is a constant. Furthermore, if f and g are (Σ, N)-monogenic, and α and β are *real* constants,, then $\alpha f + \beta g$ is (Σ, N)-monogenic, and

$$[\alpha f + \beta g]^{[1]} = \alpha f^{[1]} + \beta g^{[1]}.$$

We also have

THEOREM 3.1. *If the function* $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ *is* (Σ, N)*-monogenic and possesses a* (Σ, N)*-derivative* $f^{[1]}(z)$ *in a domain* D*, then* $f^{[1]}(z)$ *is* (Σ', N)*-monogenic in* D.

Proof. From (3.5), it follows that

$$a^{[1]}{}_{2k,X^*} = a_{2k+1,YX^*}, \qquad a^{[1]}{}_{2k+1,Y} = a_{2k+1,X^*Y},$$

$$a^{[1]}{}_{2k,Y^*} = a_{2k,XY^*}, \qquad a^{[1]}{}_{2k+1,X} = a_{2k+2,Y^*X} - a_{0Y^*X}\binom{N}{k+1},$$

$(k = 0, \cdots, N-1)$, and since

$$a^{[1]}{}_{2N-1,X} = -a_{0,Y^*X},$$

we obtain

$$\begin{aligned} &a^{[1]}{}_{2k,X^*} = a^{[1]}{}_{2k+1,Y}, \\ &a^{[1]}{}_{2k,Y^*} = a^{[1]}{}_{2k-1,X} - a^{[1]}{}_{2N-1,X}\binom{N}{k}, \end{aligned} \tag{3.6}$$

$(k = 0, \cdots, N-1)$, which is $E(\Sigma', N)$, by (1.14).

Next, we proceed to the definition of higher (Σ, N)-derivatives of a (Σ, N)-monogenic function. If the function $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic in a domain D and all the a_i belong to class $C^{(m)}$, $m \geq 2$, then the higher (Σ, N)-derivatives $f^{[n]}(z) = \sum_{i=0}^{2N-1} a_i{}^{[n]} j_N{}^i$ are defined by

$$f^{[0]}(z) = f(z),$$

$$f^{[n]}(z) = \begin{cases} \dfrac{d_{(\Sigma,N)} f^{[n-1]}(z)}{d_{(\Sigma,N)} z} & \text{if } n \text{ is odd}, \\ \dfrac{d_{(\Sigma',N)} f^{[n-1]}(z)}{d_{(\Sigma',N)} z} & \text{if } n \text{ is even}, \end{cases} \tag{3.7}$$

$(n = 1, 2, \cdots, m-1)$. Obviously, $f^{[2n]}(z)$ is (Σ, N)-monogenic and $f^{[2n+1]}(z)$ is (Σ', N)-monogenic.

B. The (Σ, N)-integral of a continuous function $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ over a rectifiable curve C is defined by

$$\int_C f(z)d_{(\Sigma,N)}z = \sum_{i=0}^{2N-1} A_i j_N^i, \tag{3.8}$$

where

$$A_{2k} = \int_C a_{2k}dX^* + [a_{2k-1} - a_{2N-1}\binom{N}{k}]dY^*$$

$$= \int_C \sigma_2 a_{2k}dx + \tau_2[a_{2k-1} - a_{2N-1}\binom{N}{k}]dy, \tag{3.9}$$

$$A_{2k+1} = \int_C a_{2k+1}dX + a_{2k}dY$$

$$= \int_C (1/\sigma_1)a_{2k+1}dx + (1/\tau_1)a_{2k}dy,$$

$(k = 0, \cdots, N-1)$.

Introducing the ordinary complex-valued functions

$$h_{2k}(z) = \sigma_2 a_{2k} - i\tau_2[a_{2k-1} - a_{2N-1}\binom{N}{k}], \tag{3.10}$$
$$h_{2k+1}(z) = (1/\tau_1)a_{2k} + i(1/\sigma_1)a_{2k+1},$$

$(k = 0, \cdots, N-1)$, and recalling the expression for $\int_C g(z)dz$, where $dz = dx + idy$, (the ordinary complex integral over C of the continuous function $g(z) = s + it$), equation (3.8) may be rewritten in the form

$$\int_C f(z)d_{(\Sigma,N)}z = \sum_{k=0}^{N-1}\{Re[\int_C h_{2k}(z)dz]\}j_N^{2k} \tag{3.11}$$

$$+ \sum_{k=0}^{N-1}\{Im[\int_C h_{2k+1}(z)dz]\}j_N^{2k+1}.$$

From (3.8) and (3.9), it follows that if $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ and $g(z) = \sum_{i=0}^{2N-1} b_i j_N^i$ are continuous on C and α and β are *real* constants, then

$$\int_C (\alpha f + \beta g)d_{(\Sigma,N)}z = \alpha \int_C f d_{(\Sigma,N)}z + \beta \int_C g d_{(\Sigma,N)}z.$$

Also, over C

$$\int_a^b f d_{(\Sigma,N)}z + \int_b^c f d_{(\Sigma,N)}z = \int_a^c f d_{(\Sigma,N)}z.$$

Let l be the length of C; A, a constant such that for all points of C, $|x| \leqq A$, $|y| \leqq A$, and $m_\Sigma(A)$ the non-decreasing function defined by $m_\Sigma(A) = \max_{i=1,2} \max_{l=1} \max_{|t|\leqq A} \{\sigma_i(t)^l, \tau_i(t)^l\}$. The following inequality holds

$$\| \int_C f d_{(\Sigma,N)} z \| \leq (2^N + 2N) m_\Sigma(A) \cdot \max_C \| f \| \cdot l. \tag{3.12}$$

For, from (3.11)

$$\| \int_C f(z) d_{(\Sigma,N)} z \| \leq (\sum_{k=0}^{N-1} \max_C | h_{2k} | + \sum_{k=0}^{N-1} \max_C | h_{2k+1} |) l,$$

while from (3.10), letting $M = \max_C \| f \|$ and $m = m_\Sigma(A)$

$$| h_{2k+1} | \leq 2mM,$$

$(k = 0, \cdots, N-1)$, and

$$| h_0 | \leq 2mM,$$
$$| h_{2k} | \leq mM(2 + \binom{N}{k}).$$

THEOREM 3.2. *If* $f(z) = \sum_{l=0}^{2N-1} a_l j_N{}^l$ *is* (Σ, N)*-monogenic in a closed simply connected domain* D *bounded by a rectifiable curve* C *then*

$$\int_C f(z) d_{(\Sigma,N)} z = 0. \tag{3.13}$$

Proof. This follows immediately upon observing that the line integrals in (3.9) are all zero in view of (3.1) or (3.3).

In view of the independence of the path asserted by Theorem 3.2, we may consider the function

$$F(z) = \int_{z_0}^{z} f d_{(\Sigma,N)} z = \sum_{i=0}^{2N-1} A_i j_N{}^i,$$

where $f = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic in D. It is easily shown that

THEOREM 3.3. *The* (Σ, N)*-integral of a* (Σ, N)*-monogenic function is* (Σ', N)*-monogenic.*

For, from (3.9)

$$\begin{aligned} A_{2k,X^\bullet} &= a_{2k}, & A_{2k+1,Y} &= a_{2k}, \\ A_{2k,Y^\bullet} &= a_{2k-1} - a_{2N-1}\binom{N}{k}, & A_{2k+1,X} &= a_{2k+1}, \end{aligned} \tag{3.14}$$

$(k = 0, \cdots, N-1)$, and since

$$A_{2N-1,X} = a_{2N-1},$$

we obtain $E(\Sigma', N)$ from (3.14).

From (3.14) and Theorem 3.3, by a simple induction, we may prove the following corollary

COROLLARY 3.3. *Let* $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ *be* (Σ, N)*-monogenic in a simply connected domain* D *and define*

$$F_0(z) = f(z),$$

$$F_n(z) = \begin{cases} \int_a^z F_{n-1}(\xi)\, d_{(\Sigma',N)}\xi & \text{if } n \text{ is odd,} \\ \int_a^z F_{n-1}(\xi)\, d_{(\Sigma',N)}\xi & \text{if } n \text{ is even,} \end{cases} \tag{3.15}$$

for $n \geqq 1$, *where* a *is a fixed point of* D. *Then* $F_n(z) = \sum_{i=0}^{2N-1} A_i{}^{(n)} j_N{}^i$ *is* (Σ, N)*-monogenic for* n *even and* (Σ', N)*-monogenic for* n *odd. Furthermore, the functions* $A_i{}^{(n)}$ *belong to class* $C^{(n)}$ *and for* $q = 0, 1, \cdots$

$$\begin{aligned} &\frac{\partial^{2q} A_{2k}{}^{(2q)}}{(\partial X^* \partial X)^q} = a_{2k}, \qquad &&\frac{\partial^{2q} A_{2k+1}{}^{(2q)}}{(\partial X \partial X^*)^q} = a_{2k+1}, \\ &\frac{\partial^{2q+1} A_{2k}{}^{(2q+1)}}{\partial X^* (\partial X \partial X^*)^q} = a_{2k}, \qquad &&\frac{\partial^{2q+1} A_{2k+1}{}^{(2q+1)}}{\partial X (\partial X^* \partial X)^q} = a_{2k+1}, \end{aligned} \tag{3.16}$$

$(k = 0, \cdots, N-1)$.

4. Contraction. In the preceding section, we defined (Σ, N)- and (Σ', N)-differentiation and integration. By repeated application of these processes, we can obtain new solutions of the system $E(\Sigma, N)$ from any given solution of this system. Here, we shall be concerned with a process which enables us to find solutions of the systems $E(\Sigma, M)$, $1 \leqq M \leqq N-1$, given a solution of the system $E(\Sigma, N)$.

LEMMA 4.1. *If* $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ *is* (Σ, N)*-monogenic in a domain* D *then* $\phi(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}{}^i$, *where*

$$\begin{aligned} \alpha_{2k} &= a_{2k} - \binom{N-1}{k} a_{2N-2}, \\ \alpha_{2k+1} &= a_{2k+1} - \binom{N-1}{k} a_{2N-1}, \end{aligned} \tag{4.1}$$

$(k = 0, \cdots, N-2)$, *is* $(\Sigma, N-1)$*-monogenic in* D.

Proof. This is easily verified using (4.1) and the binomial identity

$$\binom{N-1}{k-1} + \binom{N-1}{k} = \binom{N}{k}.$$

THEOREM 4.1. *If the function* $f_N(z) = \sum_{k=0}^{N-1} a_{2k} j_N{}^{2k} + \sum_{k=0}^{N-1} a_{2k+1} j_N{}^{2k+1}$ *is* (Σ, N)*-monogenic in a domain* D *then for each* $i = 0, \cdots, N-1$ *the function*

$$\begin{aligned} f_{N-i}(z) &= \sum_{k=0}^{N-1} a_{2k} j_{N-i}{}^{2k} + \sum_{k=0}^{N-1} a_{2k+1} j_{N-i}{}^{2k+1} \\ &= \sum_{k=0}^{N-i-1} \Big[a_{2k} + \sum_{\nu=N-i}^{N-1} \Big\{ (-1)^{N-i+\nu+1} \binom{\nu}{k} \binom{\nu-k-1}{N-i-k-1} a_{2\nu} \Big\} \Big] j_{N-i}{}^{2k} \\ &\quad + \sum_{k=0}^{N-i-1} \Big[a_{2k+1} + \sum_{\nu=N-i}^{N-1} \Big\{ (-1)^{N-i+\nu+1} \binom{\nu}{k} \binom{\nu-k-1}{N-i-k-1} a_{2\nu+1} \Big\} \Big] j_{N-i}{}^{2k+1}, \end{aligned} \tag{4.2}$$

is $(\Sigma, N-i)$*-monogenic in* D.

A sketch of the proof is as follows. First, using (2.28), we verify that the formal substitution of j_{N-i} for j_N in $f_N(z)$ yields the expression on the extreme right of (4.2). To prove that $f_{N-i}(z)$ is $(\Sigma, N-i)$-monogenic, we proceed by induction, employing Lemma 4.1 and the binomial identity

$$\begin{aligned} &\binom{\nu}{k}\binom{\nu-k-1}{N-i-k-1} + \binom{\nu}{k}\binom{\nu-k-1}{N-i-k-2} \\ &\qquad = \binom{N-i-1}{k}\binom{\nu}{N-i-1}. \end{aligned}$$

In particular, we observe that taking $i = N-1$ shows that

$$f_1(z) = \sum_{k=0}^{N-1} (-1)^k a_{2k} + j_1 \sum_{k=0}^{N-1} (-1)^k a_{2k+1},$$

is always $(\Sigma, 1)$-monogenic.

5. The partial differential equations of the $2N$th order. The main purpose of this section is to show that if $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic in a domain D, then the real functions a_{2k} $(k = 0, \cdots, N-1)$ satisfy in D the partial differential equation of the $2N$-th order

$$\left(\frac{\partial^2}{\partial X^* \partial X} + \frac{\partial^2}{\partial Y \partial Y^*} \right)^N u = 0,$$

and the real functions a_{2k+1} $(k = 0, \cdots, N-1)$ satisfy in D the partial differential equation of the $2N$-th order

$$\left(\frac{\partial^2}{\partial X \partial X^*} + \frac{\partial^2}{\partial Y^* \partial Y}\right)^N u = 0.$$

In addition, it will be shown that all the theorems of Section **3** hold under the sole assumption that the functions a_i belong to class $C^{(1)}$.

LEMMA 5.1. *If* $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ *is* (Σ, N)*-monogenic in a domain* D *and the* a_i *belong to class* $C^{(3)}$ *in* D, *then the function* $g(z) = \sum_{i=0}^{2N-1} b_i j_N^i$, *where*

$$b_{2k} = a_{2k+1, X^*Y} = (\tau_1/\sigma_2) a_{2k+1, xy},$$

$$b_{2k+1} = a_{2k+2, XY^*} - a_{0, XY^*}\left(\frac{N}{k+1}\right) = (\sigma_1/\tau_2)\left[a_{2k+2, xy} - a_{0, xy}\left(\frac{N}{k+1}\right)\right],$$

$(k = 0, \cdots, N-1)$, *is also* (Σ, N)*-monogenic in* D.

Proof. It is readily verified, from (3.7) and (3.5), that $g(z) = f^{[2]}(z)$. That $g(z)$ is (Σ, N)-monogenic follows from the remark immediately after equation (3.7).

THEOREM 5.1. *If* $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ *is* (Σ, N)*-monogenic in a domain* D *and the* a_i *belong to class* $C^{(2N)}$ *in* D, *then*

$$\begin{aligned} L^N a_{2k} &= 0, \\ M^N a_{2k+1} &= 0, \end{aligned} \tag{5.1}$$

$(k = 0, \cdots, N-1)$, *in* D, *where* L *and* M *are the linear differential operators defined by*

$$\begin{aligned} Lu &\equiv (\tau_1/\sigma_2)\left[(\sigma_1 u_x/\tau_1)_x + (\sigma_2 u_y/\tau_2)_y\right] = U_{XX^*} + U_{Y^*Y}, \\ Mu &\equiv (\sigma_1/\tau_2)\left[(\tau_2 u_x/\sigma)_x + (\tau_1 u_y/\sigma_1)_y\right] = U_{X^*X} + U_{YY^*}, \end{aligned} \tag{5.2}$$

Proof. We proceed by induction. $E(\Sigma, 1)$ is the system of equations considered by Bers and Gelbart (1)

$$\begin{aligned} \sigma_1 a_{0,x} &= \tau_1 a_{1,y} \\ \sigma_2 a_{0,y} &= -\tau_2 a_{1,x}, \end{aligned} \qquad \text{or} \qquad \begin{aligned} a_{0,X} &= a_{1,Y}, \\ a_{0,Y^*} &= -a_{1,X^*}, \end{aligned}$$

Assuming that a_1 and a_0 belong to class $C^{(2)}$, and eliminating a_1 and a_0 in turn, we obtain

$$\begin{aligned} La_0 &= 0, \\ Ma_1 &= 0, \end{aligned}$$

and the conclusion of the theorem has been verified for $N = 1$.

Suppose now that the conclusion is true for $N-1 \geqq 1$ and let $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ be (Σ, N)-monogenic, where the a_i belong to class $C^{(2N)}$. We have $E(\Sigma, N)$

$$\begin{aligned} a_{2k,X} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= a_{2k-1,X^*} - a_{2N-1,X^*} \binom{N}{k}, \end{aligned} \tag{5.3}$$

$(k = 0, \cdots, N-1)$. By Lemma 4.1, the function $\phi(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}{}^i$ is $(\Sigma, N-1)$-monogenic, where

$$\begin{aligned} \alpha_{2k} &= a_{2k} - \binom{N-1}{k} a_{2N-2}, \\ \alpha_{2k+1} &= a_{2k+1} - \binom{N-1}{k} a_{2N-1}, \end{aligned} \tag{5.4}$$

$(k = 0, \cdots, N-2)$. Consequently, by the induction hypothesis, since the α_i belong to the class $C^{(2N)}$ in D

$$\begin{aligned} L^{N-1}\alpha_{2k} &= 0, \\ M^{N-1}\alpha_{2k+1} &= 0, \end{aligned} \tag{5.5}$$

$(k = 0, \cdots, N-2)$. Since L and M are linear operators, (5.4) and (5.5) imply

$$\begin{aligned} L^{N-1}a_{2k} &= \binom{N-1}{k} L^{N-1}a_{2N-2}, \\ M^{N-1}a_{2k+1} &= \binom{N-1}{k} M^{N-1}a_{2N-1}, \end{aligned} \tag{5.6}$$

$(k = 0, \cdots, N-2)$. It is obvious then that to verify (5.1) it suffices to show that $L^N a_{2N-2}$ and $M^N a_{2N-1}$ are both zero.

From (5.3), for $k = 0$,

$$\begin{aligned} a_{0,X} &= a_{1,Y}, \\ a_{0,Y^*} &= -a_{2N-1,X^*}, \end{aligned} \tag{5.7}$$

Using the definition of L, (5.2) and (5.4), equation (5.7) yields

$$La_0 = \alpha_{1,X^*Y}, \tag{5.8}$$

since

$$\begin{aligned} La_0 &= a_{0,XX^*} + a_{0,Y^*Y} \\ &= a_{1,YX^*} - a_{2N-1,X^*Y} = \left[a_1 - \binom{N-1}{0} a_{2N-1}\right]_{X^*Y}. \end{aligned}$$

From (3.4) for $k=0$

$$\begin{aligned} a_{1,X^\bullet} &= a_{2,Y^\bullet} - a_{0,Y^\bullet}(N-1), \\ a_{1,Y} &= a_{0,X^\bullet}. \end{aligned} \tag{5.9}$$

Using the definition of M, (5.2), and (5.4), equation (5.9) yields

$$Ma_1 = \alpha_{2,XY^\bullet} - \binom{N-1}{1}\alpha_{0,XY^\bullet}, \tag{5.10}$$

since

$$\begin{aligned} Ma_1 &= a_{1,X^\bullet X} + a_{1,YY^\bullet} \\ &= a_{2,Y^\bullet X} - a_{0,Y^\bullet X}(N-1) + a_{0,XY^\bullet} \\ &= [a_2 - \binom{N-1}{1} a_0]_{XY^\bullet}, \end{aligned}$$

and from (5.4)

$$\begin{aligned} \alpha_2 - \binom{N-1}{1}\alpha_0 &= [a_2 - \binom{N-1}{1} a_{2N-2}] - \binom{N-1}{1}[a_0 \\ &\quad - \binom{N-1}{0} a_{2N-2}] = a_2 - \binom{N-1}{1} a_0. \end{aligned}$$

On the other hand, since $N \geqq 2$, we may apply Lemma 5.1 to $\phi(z)$ to prove that the function $\phi^{[2]}(z) = \sum_{i=0}^{2N-3} \alpha_i{}^{[2]} j_{N-1}{}^i$ is $(\Sigma, N-1)$-monogenic, where

$$\begin{aligned} \alpha_{2k}{}^{[2]} &= \alpha_{2k+1,X^\bullet Y}, \\ \alpha_{2k+1}{}^{[2]} &= \alpha_{2k+2,XY^\bullet} - \alpha_{0,XY^\bullet}\binom{N-1}{k+1}, \end{aligned} \tag{5.11}$$

$(k=0,\cdots,N-2)$. Consequently, by the induction hypothesis, inasmuch as the α_i belong to class $C^{(2N-2)}$, we have

$$\begin{aligned} L^{N-1}\alpha_{2k}{}^{[2]} &= 0, \\ M^{N-1}\alpha_{2k+1}{}^{[2]} &= 0, \end{aligned} \tag{5.12}$$

$(k=0,\cdots,N-2)$.

But a comparison of (5.8), and (5.10) with (5.11) gives

$$\begin{aligned} La_0 &= \alpha_0{}^{[2]}, \\ Ma_1 &= \alpha_1{}^{[2]}, \end{aligned} \tag{5.13}$$

and, in view of (5.12), (5.13) yields

$$\begin{aligned} L^N a_0 &= 0, \\ M^N a_1 &= 0. \end{aligned} \tag{5.14}$$

The desired conclusion, (5.1), follows from (5.6) and (5.14).

It is now possible to remove most of the differentiability assumptions made on the a_i in the hypotheses of several theorems. It will be recalled that σ_1, σ_2, τ_1, and τ_2 are positive analytic functions of their respective real variables. Hence, the operators L and M are of elliptic type with analytic coefficients and any function u of class $C^{(2N)}$ satisfying either $L^N u = 0$ or $M^N u = 0$ for a fixed $N = 1, 2, \cdots$, is necessarily analytic. Let $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ be (Σ, N)-monogenic in a simply connected domain D and consider the function

$$F_{2N}(z) = \sum_{i=0}^{2N-1} A_i{}^{(2N)} j_N{}^i,$$

(the functions $F_n(z)$ were defined in (3.15)). From Corollary 3.3, it follows that $F_{2N}(z)$ is (Σ, N)-monogenic and that the $A_i{}^{(2N)}$ belong to class $C^{(2N)}$. Hence, from Theorem 5.1

$$L^N A_{2k}{}^{(2N)} = 0,$$

$$M^N A_{2k+1}{}^{(2N)} = 0,$$

$(k = 0, \cdots, N-1)$, and consequently the $A_i{}^{(2N)}$ are analytic. But from (3.16)

$$\frac{\partial^{2N} A_{2k}{}^{(2N)}}{(\partial X^* \partial X)^N} = a_{2k}, \qquad \frac{\partial^{2N} A_{2k+1}{}^{(2N)}}{(\partial X \partial X^*)^N} = a_{2k+1},$$

$(k = 0, \cdots, N-1)$, which shows that $a_i(x, y)$ is a linear combination, with analytic coefficients, of $A_i{}^{(2N)}(x, y)$ and its partial derivatives with respect to x up to the $2N$-th order. Thus the functions $a_i(x, y)$ are analytic functions of x and y and we have the

THEOREM 5.2. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic in a domain D, then the functions $a_i(x, y)$ are analytic functions of x and y in D.*

Using this result, we see that unlimited successive (Σ, N)- and (Σ', N)-differentiations of a (Σ, N)-monogenic function are possible.

Theorem 5.1 may now be restated in the form

THEOREM 5.3. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic in a domain D, then*

$$L^N a_{2k} = 0,$$

$$M^N a_{2k+1} = 0,$$

$(k = 0, \cdots, N-1)$, in D.

We remark that (Σ, N)-integration and (Σ', N)-differentiation, as well as (Σ, N)-differentiation and (Σ', N)-integration, are inverse processes. That is, if $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N)-monogenic, then

$$\frac{d_{(\Sigma',N)}}{d_{(\Sigma',N)}z} \int_{z_0}^{z} f(\zeta) d_{(\Sigma,N)}\zeta = f(z), \tag{5.15}$$

and

$$\int_{z_0}^{z} f^{[1]}(\zeta) d_{(\Sigma',N)}\zeta = f(z) - f(z_0), \tag{5.16}$$

Next, we prove the analogue of Morera's theorem.

THEOREM 5.4. *If* $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ *is continuous in a simply connected domain* D *and for every simple closed rectifiable curve* C *in* D, $\int_C f(z) d_{(\Sigma,N)} z = 0$, *then* $f(z)$ *is* (Σ, N)*-monogenic in* D.

Proof. Consider the function

$$F(z) = \int_{z_0}^{z} f d_{(\Sigma,N)} z = \sum_{i=0}^{2N-1} A_i j_N^i.$$

By (3.14), as in the proof of Theorem 3.3, we see that $f(z)$ is (Σ', N)-monogenic. By (3.14) and (3.5), $\frac{d_{(\Sigma',N)} F}{d_{(\Sigma',N)} z} = f(z)$.

THEOREM 5.5. *If a series of* (Σ, N)*-monogenic functions* $\sum_{n=0}^{\infty} f_n(z)$, *where* $f_n(z) = \sum_{n=0}^{\infty} a_{ni} j_N^i$, *converges uniformly to* $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ *in a domain* D, *then* $f(z)$ *is* (Σ, N)*monogenic in* D.

6. The associated matrices. The system $E(\Sigma, N)$ has

$$\Sigma = \begin{Vmatrix} \sigma_1 & \tau_1 \\ \sigma_2 & \tau_2 \end{Vmatrix},$$

as the matrix of its coefficients. Together with $E(\Sigma, N)$, we shall be led to consider systems whose coefficient matrices are

$$\Sigma' = \begin{Vmatrix} 1/\sigma_2 & \tau_1 \\ 1/\sigma_1 & \tau_2 \end{Vmatrix}, \quad \Sigma_{,} = \begin{Vmatrix} \sigma_1 & 1/\tau_2 \\ \sigma_2 & 1/\tau_1 \end{Vmatrix}, \quad \Sigma_{,}' = \begin{Vmatrix} 1/\sigma_2 & 1/\tau_2 \\ 1/\sigma_1 & 1/\tau_1 \end{Vmatrix}. \tag{6.1}$$

Obviously,

$$\Sigma'' \equiv \Sigma_{,,} \equiv \Sigma_{,,}'' \equiv \Sigma.$$

It has already been shown that the (Σ, N)-integral and the (Σ, N)-derivative of a (Σ, N)-monogenic function are both (Σ', N)-monogenic functions. We now prove two theorems concerning the relations between (Σ, N)-, (Σ', N)-, $(\Sigma_{,}, N)$-, and $(\Sigma_{,}', N)$-monogenic functions.

THEOREM 6.1. *If* $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ *is* (Σ, N)*-monogenic, then* $g(z) \equiv j_N f(z)$ *is* $(\Sigma_{,}', N)$*-monogenic.*

Letting $g(z) = \sum_{i=0}^{2N-1} b_i j_N^i$ and using Theorem 2.6, the result is immediate.

COROLLARY 6.1. *If* $f(z)$ *is* (Σ, N)*-monogenic, then* $j_N^{2k} f(z)$ *is* (Σ, N)*-monogenic and* $j_N^{2k+1} f(z)$ *is* $(\Sigma_{,}', N)$*-monogenic, where* k *is any integer.*

The proof follows for positive k by repeated application of Theorem 6.1, and then for negative k by using Theorem 2.5.

THEOREM 6.2. *If* $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ *is* (Σ, N)*-monogenic, then*

$$\frac{d_{(\Sigma_{,}', N)}[j_N f]}{d_{(\Sigma_{,}', N)} z} = j_N \frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z}, \tag{6.3}$$

and

$$\int_{z_0}^{z} j_N f d_{(\Sigma_{,}', N)} z = j_N \int_{z_0}^{z} f d_{(\Sigma, N)} z. \tag{6.4}$$

Proof. (6.3) is a consequence of the definition of the (Σ, N)-derivative, equation (3.5), and Theorem 6.1. As regards (6.4), both sides of the proposed equality are $(\Sigma_{,}, N)$-monogenic functions. In addition, both functions vanish at z_0 and, by (6.3), the $(\Sigma_{,}, N)$-derivative of their difference vanishes identically.

COROLLARY 6.2. *If* $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ *is* (Σ, N)*-monogenic, then*

$$\frac{d_{(\Sigma, N)}[j_N^{2k} f]}{d_{(\Sigma, N)} z} = j_N^{2k} \frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z}, \tag{6.5}$$

$$\int_{z_0}^{z} j_N^{2k} f d_{(\Sigma, N)} z = j_N^{2k} \int_{z_0}^{z} f d_{(\Sigma, N)} z, \tag{6.6}$$

$$\frac{d_{(\Sigma_{,}', N)}[j_N^{2k+1} f]}{d_{(\Sigma_{,}', N)} z} = j_N^{2k+1} \frac{d_{(\Sigma, N)} f}{d_{(\Sigma, N)} z}, \tag{6.7}$$

$$\int_{z_0}^{z} j_N^{2k+1} f d_{(\Sigma_{,}', N)} z = j_N^{2k+1} \int_{z_0}^{z} f d_{(\Sigma, N)} z, \tag{6.8}$$

where k *is any integer.*

7. The formal powers.

A. Any hypercomplex constant is a (Σ, N)-monogenic function; hence, we may construct (Σ, N)-monogenic functions by successive (Σ, N)- and (Σ', N)-integrations of a constant. Accordingly, we define [see Bers and Gelbart (1), Section **4**]

$$
\begin{aligned}
&a \cdot Z^{(0)}(N; z_0; z) = a \cdot \tilde{Z}^{(0)}(N; z_0; z) = a, \\
&a \cdot Z^{(n)}(N; z_0; z) = n \int_{z_0}^{z} (a \cdot \tilde{Z}^{(n-1)}(N; z_0; z))\, d_{(\Sigma', N)} z, \\
&a \cdot \tilde{Z}^{(n)}(N; z_0; z) = n \int_{z_0}^{z} (a \cdot Z^{(n-1)}(N; z_0 z))\, d_{(\Sigma, N)} z,
\end{aligned}
\tag{7.1}
$$

for $n \geqq 1$, where $a = \sum_{i=0}^{2N-1} \alpha_i j_N^i$ is a hypercomplex constant and the point z_0 is fixed. In this formula $a \cdot Z$ is *one* symbol and does not denote ordinary multiplication of a by Z, since no meaning has been attached to Z by itself as yet. In addition, we have

$$
a \cdot Z^{(n)}(N; z_0; z_0) = a \cdot \tilde{Z}^{(n)}(N; z_0; z_0) = 0 \tag{7.2}
$$

and

$$
\begin{aligned}
&\frac{d_{(\Sigma, N)}(a \cdot Z^{(n)}(N; z_0; z))}{d_{(\Sigma, N)} z} = n a \cdot \tilde{Z}^{(n-1)}(N; z_0; z), \\
&\frac{d_{(\Sigma', N)}(a \cdot \tilde{Z}^{(n)}(N; z_0; z))}{d_{(\Sigma', N)} z} = n a \cdot Z^{(n-1)}(N; z_0; z).
\end{aligned}
\tag{7.3}
$$

In this section, we shall keep N fixed. To simplify the writing, we set

$$
1 \cdot Z^{(n)}(N; z_0; z) \equiv Z^{(n)}(z_0; z), \qquad a \cdot Z^{(n)}(N; z_0; z) \equiv a \cdot Z^{(n)}(z_0; z),
$$

and similarly for the $\tilde{Z}^{(n)}$'s. Also, we shall write Σ-monogenic instead of (Σ, N)-monogenic, and use j in place of j_N.

At first glance, the iterated Σ- and Σ'-integration of the $2N$ constants $1, j, j^2, \cdots j^{2N-1}$ might be expected to yield $2N$ different functions as the result of each integration operation. However, each time, essentially only two new functions appear, because

$$
\begin{aligned}
&j^{2k} \cdot Z^{(n)}(z_0; z) = j^{2k} Z^{(n)}(z_0; z), \\
&j^{2k+1} \cdot Z^{(n)}(z_0; z) = j^{2k}[j \cdot Z^{(n)}(z_0; z)],
\end{aligned}
\tag{7.4}
$$

and

$$
\begin{aligned}
&j^{2k} \cdot \tilde{Z}^{(n)}(z_0; z) = j^{2k} \tilde{Z}^{(n)}(z_0; z), \\
&j^{2k+1} \cdot \tilde{Z}^{(n)}(z_0; z) = j^{2k}[j \cdot \tilde{Z}^{(n)}(z_0; z)]
\end{aligned}
\tag{7.5}
$$

where $k = 0, 1, 2, \cdots$. Obviously, (7.4) and (7.5) hold for $n = 0$. And if they hold for $n-1$, then they also hold for n, since in view of (7.1) and (6.6)

$$\begin{aligned} j^{2k} \cdot Z^{(n)}(z_0; z) &= n \int_{z_0}^{z} (j^{2k} \cdot \tilde{Z}^{(n-1)}(z_0; z)) d_{\Sigma'} z \\ &= n \int_{z_0}^{z} j^{2k} \tilde{Z}^{(n-1)}(z_0; z) d_{\Sigma'} z \\ &= j^{2k} n \int_{z_0}^{z} \tilde{Z}^{(n-1)}(z_0; z) d_{\Sigma'} z \\ &= j^{2k} Z^{(n)}(z_0; z), \end{aligned}$$

which verifies the first equation of (7.4). A similar reasoning yields the second equation of (7.4). The two equations of (7.5) may be derived analogously, or obtained at once by noticing that if the formal powers formed with respect to the matrix Σ' are denoted by $a \cdot Z'^{(n)}$, etc., we have

$$\begin{aligned} a \cdot Z'^{(n)}(z_0; z) &= a \cdot \tilde{Z}^{(n)}(z_0; z), \\ a \cdot \tilde{Z}'^{(n)}(z_0; z) &= a \cdot Z^{(n)}(z_0; z), \end{aligned} \tag{7.6}$$

and that (7.4) written with respect to Σ' is (7.5).

Clearly, for $a = \sum_{i=0}^{2N-1} \alpha_i j^i$, the α_i being real constants,

$$a \cdot Z^{(n)}(z_0; z) = \sum_{i=0}^{2N-1} \alpha_i [j^i \cdot Z^{(n)}(z_0; z)], \tag{7.7}$$

and similarly for $a \cdot \tilde{Z}^{(n)}(z_0; z)$.

With the aid of (7.4), we obtain, from (7.7), the formula

$$\begin{aligned} [\sum_{i=0}^{2N-1} \alpha_i j^i] \cdot Z^{(n)}(z_0; z) &= [\sum_{k=0}^{N-1} \alpha_{2k} j^{2k}] Z^{(n)}(z_0; z) \\ &\quad + [\sum_{k=0}^{N-1} \alpha_{2k+1} j^{2k}][j \cdot Z^{(n)}(z_0; z)], \end{aligned} \tag{7.8}$$

which gives $a \cdot Z^{(n)}$ explicitly, once $Z^{(n)}$ and $j \cdot Z^{(n)}$ are known.

We call $a \cdot Z^{(n)}$ the "formal product" of "a" and the "formal power" $Z^{(n)}$.

B. If $f(z) = \sum_{i=0}^{2N-1} a_i j^i$, $a_i = a_i(x, y)$, is Σ-monogenic, then the a_i satisfy $E(\Sigma, N)$. Consequently, the formal powers $a \cdot Z^{(n)}(z_0; z)$ furnish us an infinite number of particular solutions of the system $E(\Sigma, N)$. In order to

further identify these particular solutions, and to obtain formulas which are formally equivalent to the binomial formula for $a(z-z_0)^n$, we use the following real functions (1): [See (1.10)]

$$X^{(0)}(x_0;x) = X^{*(0)}(x_0;x) = Y^{(0)}(y_0;y) = Y^{*(0)}(y_0;y) \equiv 1$$

$$(7.9)\qquad X^{(n)}(x_0;x) = \begin{cases} n\int_{x_0}^{x} X^{(n-1)}dX & \text{if } n \text{ is odd,} \\ n\int_{x_0}^{x} X^{(n-1)}dX^* & \text{if } n \text{ is even,} \end{cases}$$

$$X^{*(n)}(x_0;x) = \begin{cases} n\int_{x_0}^{x} X^{*(n-1)}dX^* & \text{if } n \text{ is odd,} \\ n\int_{x_0}^{x} X^{*(n-1)}dX & \text{if } n \text{ is even,} \end{cases}$$

$$Y^{(n)}(y_0;y) = \begin{cases} n\int_{y_0}^{y} Y^{(n-1)}dY & \text{if } n \text{ is odd,} \\ n\int_{y_0}^{y} Y^{(n-1)}dY^* & \text{if } n \text{ is even,} \end{cases}$$

$$Y^{*(n)}(y_0;y) = \begin{cases} n\int_{y_0}^{y} Y^{*(n-1)}dY^* & \text{if } n \text{ is odd,} \\ n\int_{y_0}^{y} Y^{*(n-1)}dY & \text{if } n \text{ is even.} \end{cases}$$

In other words:

$$X^{(n)}(x_0;x) = n!\int_{x_0}^{x} 1/\sigma_1 \int_{x_0}^{\sigma_1}\cdots\int_{x_0}^{\sigma_2}\int_{x_0}^{\sigma_2} dx^n/\sigma_1 \qquad (n \text{ integrals, } n \text{ odd})$$

$$X^{(n)}(x_0;x) = n!\int_{x_0}^{x} \sigma_2 \int_{x_0}^{\sigma_2} 1/\sigma_1\cdots\int_{x_0}^{\sigma_2}\int_{x_0}^{\sigma_2} dx^n/\sigma_1 \qquad (n \text{ integrals, } n \text{ even})$$

and similarly for $X^{*(n)}$, $Y^{(n)}$, $Y^{*(n)}$. If $x_0 = 0$ (or $y_0 = 0$), we write simply $X^{(n)}(x)$, $X^{*(n)}(x)$, (or $Y^{(n)}(y)$, $Y^{*(n)}(y)$). It is to be noticed that the definition of these functions is independent of N.

Making use of the functions introduced in (7.9), we shall be able to obtain an analogue of the binomial formula for $(x + j_N y)^p$, by replacing the powers of x and y in (2.31) and (2.32) by the appropriate functions of (7.9). We have, for $q = 0, 1, 2, \cdots$, in the notation of (2.33),

$$(7.10)\qquad Z^{(2q)}(z_0;z) = \sum_{i=0}^{2N-1}\Big[\sum_{l=0}^{2q} A_{i,l,2q}{}^{(N)}\, Y^{(l)}(y_0;y)\, X^{*(2q-l)}(x_0;x)\Big] j^i$$

$$(7.11)\qquad Z^{(2q+1)}(z_0;z) = \sum_{i=0}^{2N-1}\Big[\sum_{l=0}^{2q+1} A_{i,l,2q+1}{}^{(N)}\, Y^{(l)}(y_0;y)X^{(2q+1-l)}(x_0;x)\Big]j^i$$

$$(7.12)\qquad j\cdot Z^{(2q)}(z_0;z) = \sum_{i=0}^{2N-1}\Big[\sum_{l=0}^{2q} B_{i,l,2q}{}^{(N)}\, Y^{*(l)}(y_0;y)X^{(2q-l)}(x_0;x)\Big]j^i$$

$$(7.13)\qquad j\cdot Z^{(2q+1)}(z_0;z) = \sum_{i=0}^{2N-1}\Big[\sum_{l=0}^{2q+1} B_{i,l,2q+1}{}^{(N)}\, Y^{*(l)}(y_0;y)\, X^{*(2q+1-l)}(x_0;x)\Big]j^i.$$

Equation (7.10) may be verified by induction, employing the binomial identity (2.29). (7.11) is then obtained by Σ- and Σ'-differentiating $\dfrac{1}{(2q+2)(2q+1)} Z^{(2q+2)}(z_0;z)$. (7.12) and (7.13) follow immediately, after observing that if we introduce the formal powers $a\cdot Z_{,}{}'^{(n)}$, etc., which are formed with respect to the matrix $\Sigma_{,}{}'$, we have

$$j\cdot Z^{(n)}(z_0;z) = jZ_{,}{}'^{(n)}(z_0;z).$$

C. It is possible to obtain a simple inequality for the absolute value of the formal powers. We have

$$(7.14)\qquad \| \alpha\cdot Z^{(n)}(z_0;z)\| \leq 2\,\|\alpha\|\,\Gamma_N[2^{\frac{1}{2}}\Gamma_N m_\Sigma(|z_0|+|z|)\,|z-z_0|\,]^n,$$

where $\alpha = \sum_{i=0}^{2N-1}\alpha_i j^i$ is a hypercomplex constant, Γ_N is the constant occurring in (2.26), and $|z_0|^2 = x_0^2+y_0^2$, $|z|^2 = x^2+y^2$.

First, we obtain inequalities for $Z^{(n)}(z_0;z)$ and for $j\cdot Z^{(n)}(z_0;z)$. Using the non-decreasing function $m_\Sigma(A)$, defined by

$$m_\Sigma(A) = \max_{i=1,2}\ \max_{l=\pm 1}\ \max_{|t|\leqq A}\{\sigma_i(t)^l, \tau_i(t)^l\},$$

we have that [see (1), page 75]

$$(7.15)\qquad m_\Sigma{}^{-n}(|x_0|+|x|) \leq \frac{X^{(n)}(x_0;x)}{(x-x_0)^n} \leq m_\Sigma{}^n(|x_0|+|x|),$$

with similar inequalities for X^*, Y and Y^*. We also know [see (2.31) and (2.32)] that

$$(7.16)\qquad Z^{(n)}(z_0;z) = \begin{cases} \sum_{i=0}^{n}\binom{n}{i} X^{(n-i)}Y^{(i)}j^i & \text{if } n \text{ is odd,} \\ \sum_{i=0}^{n}\binom{n}{i} X^{*(n-i)}Y^{(i)}j^i & \text{if } n \text{ is even.} \end{cases}$$

$$j\cdot Z^{(n)}(z_0;z) = \begin{cases} j\sum_{i=0}^{n}\binom{n}{i} X^{*(n-i)}Y^{*(i)}j^i & \text{if } n \text{ is odd,} \\ j\sum_{i=0}^{n}\binom{n}{i} X^{(n-i)}Y^{*(i)}j^i & \text{if } n \text{ is even.} \end{cases}$$

Consequently,

$$\left.\begin{matrix} \| Z^{(n)}(z_0;z)\| \\ \| j\cdot Z^{(n)}(z_0;z)\| \end{matrix}\right\} \leqq 2^{n/2}\Gamma_N{}^n m_\Sigma{}^n(|z_0|+|z|)|z-z_0|^n, \tag{7.17}$$

since by (2.26), (2.24), (2.25), plus (2.27) with $\alpha = j$, we have

$$\| j\cdot Z^{(n)}(z_0;z)\| \leqq \begin{cases} \Gamma_N \| j\| \ \| \sum_{i=0}^{n} \binom{n}{i} X^{*(n-i)}Y^{*(i)}j^i \| \\ \Gamma_N \| j\| \ \| \sum_{i=0}^{n} \binom{n}{i} X^{(n-i)}Y^{*(i)}j^i \| \end{cases}$$

$$\leqq \Gamma_N \sum_{i=0}^{n} \binom{n}{i} m_\Sigma{}^n(|z_0|+|z|)|x-x_0|^{n-i}|y-y_0|^i\Gamma_N{}^{i}$$

$$\leqq m_\Sigma{}^n(|z_0|+|z|)\Gamma_N{}^n[|x-x_0|+|y-y_0|]^n$$

$$\leqq 2^{n/2}\Gamma_N{}^n m_\Sigma{}^n(|z_0|+|z|)|z-z_0|^n.$$

The first inequality of (7.17) may be verified similarly.

But from (7.8) and (2.27)

$$\| \alpha\cdot Z^{(n)}(z_0;z)\| \leq \Gamma_N \| \sum_{k=0}^{N-1} \alpha_{2k}j^{2k}\| \ \| Z^{(n)}(z_0;z)\|$$

$$+ \Gamma_N \| \sum_{k=0}^{N-1} \alpha_{2k+1}j^{2k}\| \ \| j\cdot Z^{(n)}(z_0;z)\| \tag{7.18}$$

$$\leq \Gamma_N \| \alpha\| [\| Z^{(n)}(z_0;z)\| + \| j\cdot Z^{(n)}(z_0;z)\|],$$

Equation (7.14) then follows from (7.17) and (7.18).

8. Null functions. There are certain (Σ, N)-monogenic functions $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ such that one or more of the real functions a_i vanish identically. This section will be devoted to the determination of all such "null functions."

THEOREM 8.1. *Let D be a domain containing the point z_0. If $f(z) = \sum_{i=0}^{q} a_i j_N{}^i$ where $0 \leq q \leq 2N-2$, is (Σ, N)-monogenic in D, then*

$$f(z) = \sum_{i=0}^{q} \alpha_i \cdot Z^{(i)}(z_0;z), \tag{8.1}$$

where

$$\alpha_i = \sum_{l=0}^{q-i} \alpha_{il} j_N{}^l, \tag{8.2}$$

$(i = 0, \cdots, q)$ are hypercomplex constants. (Conversely, any function defined by (8.1) and (8.2) is of the form $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$, where $a_{q+1} = a_{q+2} = \cdots a_{2N-1} \equiv 0$).

Proof. Given any matrix Σ, the conclusion is true for $q = 0$. For then $f(z)$ is merely a real constant. Suppose that for any matrix Σ the assertion is true for q and let

$$f(z) = \sum_{i=0}^{2N-1} a_i \mathfrak{j}_N{}^i, \qquad a_{q+2} = a_{q+3} = \cdots = a_{2N-1} \equiv 0, \tag{8.3}$$

be (Σ, N)-monogenic in D. Then

$$\begin{aligned} a_{2k,X} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= a_{2k-1,X^*}, \end{aligned} \tag{8.4}$$

$(k = 0, \cdots, N-1)$. If $q+1 = 2k$ for some k, then, in view of the first equation of (8.4),

$$a_{q+1,X} = a_{q+2,Y} = 0.$$

If $q+1 = 2k+1$ for some k, then, in view of the second equation of (8.4),

$$0 = a_{q+2,Y^*} = a_{q+1,X^*}.$$

From Definition 3.2, we see that

$$\frac{d_{(\Sigma,N)} f}{d_{(\Sigma,N)} z} = \sum_{i=0}^{q} b_i j_N{}^i. \tag{8.5}$$

The function $\dfrac{d_{(\Sigma,N)} f}{d_{(\Sigma,N)} z}$ is (Σ', N)-monogenic; hence, by virtue of the induction hypothesis, (8.5) and (8.1) give

$$\frac{d_{(\Sigma,N)} f}{d_{(\Sigma,N)} z} = \sum_{i=0}^{q} \beta_i \cdot \tilde{Z}^{(i)}(z_0; z), \tag{8.6}$$

where

$$\beta_i = \sum_{l=0}^{q-i} \beta_{il} j_N{}^l, \tag{8.7}$$

$(i = 0, \cdots, q)$ are hypercomplex constants.

Equation (8.6) may be rewritten as follows:

$$\frac{d_{(\Sigma,N)} f}{d_{(\Sigma,N)} z} = \sum_{i=1}^{q+1} \beta_{i-1} \cdot \tilde{Z}^{(i-1)}(z_0; z) = \sum_{i=1}^{q+1} i\alpha_i \cdot \tilde{Z}^{(i-1)}(z_0; z), \tag{8.8}$$

by defining

$$\alpha_i = \frac{\beta_{i-1}}{i}, \tag{8.9}$$

$(i = 1, \cdots, q+1)$. Setting

$$\alpha_{il} = \frac{\beta_{i-1,l}}{i},$$

for $(i=1,\cdots,q+1)$, $(l=0,\cdots,q-i)$, gives

$$\alpha_i = \sum_{l=0}^{q+1-i} \alpha_{il} j_N{}^l, \tag{8.10}$$

$(i=1,\cdots,q+1)$.

The function $f(z)$ of (8.3) is obtained by (Σ', N)-integrating (8.8). [This (Σ', N)-integration is possible whether D is simply connected or not, since the function given by (8.8) is (Σ', N)-monogenic at all points of the plane.] The result is

$$f(z) = \sum_{i=1}^{q+1} \alpha_i \cdot Z^{(i)}(z_0; z) + \alpha_0 = \sum_{i=0}^{q+1} \alpha_i \cdot Z^{(i)}(z_0; z), \tag{8.11}$$

where

$$\alpha_0 = \sum_{l=0}^{q+1} \alpha_{0l} j_N{}^l, \tag{8.12}$$

is a hypercomplex constant of a nature determined by the condition (8.3) on the functions a_i. The last three equations serve to verify the conclusion for $q+1$.

LEMMA 8.1. *For each $n \geqq 1$, let E_n be the set of $2n$ functions*

$$\begin{aligned} F_{2k} &= X^{(2k)}, \\ F_{2k+1} &= X^{*(2k+1)}, \end{aligned} \tag{8.13}$$

$(k=0,\cdots,n-1)$. E_n is a linearly independent set of functions on any interval $a<x<b$.

The proof is by induction.

THEOREM 8.2a. *Let D be a domain and z_0 be a point of D. If $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic in D and $a_{2k} \equiv 0$, for some fixed k, where $0 \leq k \leq N-1$, then*

$$f(z) = 1/K_{N,k} \sum_{i=0}^{2N-2} \alpha_i \cdot Z^{(i)}(z_0; z), \tag{8.14}$$

where

$$\alpha_i = \sum_{l=1}^{2N-1-i} \alpha_{il} j_N{}^l, \tag{8.15}$$

$(i=0,1,\cdots,2N-2)$ are hypercomplex constants and the $K_{N,k}$ are given by Theorem 2.7. [Conversely, any function defined by (8.14) and (8.15) is of the form $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$, where $a_{2k} \equiv 0$.]

Proof. The function

$$F(z) = \sum_{i=0}^{2N-1} A_i j_N{}^i = K_{N,k} f(z), \tag{8.16}$$

is (Σ, N)-monogenic and $A_0 = a_{2k} = 0$. In view of the second equation of $E(\Sigma, N)$, $0 = A_{0,Y^*} = A_{2N-1,X^*}$. Hence, the (Σ', N)-monogenic function $\frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z}$ has the form

$$\frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z} = \sum_{i=0}^{2N-2} A_i{}^{[1]} j_N{}^i. \tag{8.17}$$

By Theorem 8.1, we have

$$\frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z} = \sum_{i=0}^{2N-2} \beta_i \cdot \tilde{Z}^{(i)}(z_0; z), \tag{8.18}$$

where

$$\beta_i = \sum_{l=0}^{2N-2-i} \beta_{il} j_N{}^l, \tag{8.19}$$

$(i = 0, \cdots, 2N-2)$ are hypercomplex constants.

In addition, $A_0{}^{[1]} = A_{0,X} = 0$, since $A_0 = 0$. Recalling (7.8), and by virtue of (2.31) and (2.32), for $0 \leq q \leq N-1$

$$\begin{aligned} Re[\tilde{Z}^{(2q)}(z_0; z)] &= X^{(2q)}, \\ Re[j_N \cdot \tilde{Z}^{(2q)}(z_0; z)] &= 0, \end{aligned} \tag{8.20}$$

and for $0 \leq q \leq N-2$

$$\begin{aligned} Re[\tilde{Z}^{(2q+1)}(z_0; z)] &= X^{*(2q+1)}, \\ Re[j_N \cdot \tilde{Z}^{(2q+1)}(z_0; z)] &= 0, \end{aligned} \tag{8.21}$$

we obtain from (8.17) and (8.18), by equating the real part of $\frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z}$ to zero

$$A_0{}^{[1]} = \sum_{h=0}^{N-1} \beta_{2h,0} X^{(2h)} + \sum_{h=0}^{N-2} \beta_{2h+1,0} X^{*(2h+1)} \equiv 0. \tag{8.22}$$

By Lemma 8.1, (8.22) implies

$$\beta_{i,0} = 0,$$

$(i = 0, 1, 2, \cdots, 2N-2)$. Consequently, $\beta_{2N-2} = 0$ in (8.19) and equations (8.18) and (8.19) become

$$\frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z} = \sum_{i=0}^{2N-3} \beta_i \cdot \tilde{Z}^{(i)}(z_0; z), \tag{8.23}$$

and

$$\beta_i = \sum_{l=1}^{2N-2-i} \beta_{il} j_N{}^l, \tag{8.24}$$

$(i = 0, \cdots, 2N-3)$.

Let

$$\alpha_i = \frac{\beta_{i-1}}{i} = 1/i \sum_{l=1}^{2N-2-(i-1)} \beta_{i-1,l} j_N{}^l, \tag{8.25}$$

$(i = 1, \cdots, 2N-2)$, and

$$\alpha_{il} = \beta_{i-1,l}/i, \tag{8.26}$$

for $i = 1, \cdots, 2N-2$, $l = 1, \cdots, 2N-1-i$. Then, from (8.24), (8.25), and (8.26)

$$\alpha_i = \sum_{l=1}^{2N-1-i} \alpha_{il} j_N{}^l, \tag{8.27}$$

$(i = 1, \cdots, 2N-2)$.

Using (8.25) and (8.27), equation (8.23) may be rewritten thus

$$\frac{d_{(\Sigma,N)} F}{d_{(\Sigma,N)} z} = \sum_{i=1}^{2N-2} \beta_{i-1} \cdot \tilde{Z}^{(i-1)}(z_0; z) = \sum_{i=1}^{2N-2} i\alpha_i \cdot \tilde{Z}^{(i-1)}(z_0; z). \tag{8.28}$$

$F(z)$ is found by (Σ', N)-integrating (8.28).

$$F(z) = \sum_{i=1}^{2N-2} \alpha_i \cdot \tilde{Z}^{(i)}(z_0; z) + \alpha_0 = \sum_{i=0}^{2N-2} \alpha_i \cdot Z^{(i)}(z_0; z), \tag{8.29}$$

where

$$\alpha_0 = \sum_{l=1}^{2N-1} \alpha_{0l} j_N{}^l, \tag{8.30}$$

is a hypercomplex constant of a nature determined by the condition that $A_0 = 0$ in (8.16). Equations (8.29) and (8.16) yield (8.14), and equations (8.30) and (8.27) yield (8.15).

By a similar reasoning, we obtain

THEOREM 8.2b. *Let D be a domain and let z_0 be a point of D. If $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic in D and $a_{2k+1} \equiv 0$ for some fixed k, where $0 \leq k \leq N-1$, then*

$$f(z) = 1/K_{N,k}[\sum_{i=1}^{2N-2} \alpha_i \cdot Z^{(i)}(z_0; z) + \alpha_0], \tag{8.31}$$

where

$$\alpha_i = \sum_{l=2}^{2N-i} \alpha_{il} j_N{}^l, \tag{8.32}$$

($i = 1, \cdots, 2N-2$) are hypercomplex constants, $\alpha_0 = \sum_{l=0}^{2N-1} \alpha_{0l} j_N{}^l$ *is a hypercomplex constant such that* $\alpha_{01} = 0$, *and* $K_{N,k}$ *is given by Theorem 2.7. (Conversely, any function defined by (8.31) and (8.32) is of the form* $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$, *where* $a_{2k+1} \equiv 0$.)

THEOREM 8.3. *If* $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ *is* (Σ, N)*-monogenic in a domain D and one of the functions* a_i *vanishes identically in D, then* $f(z)$ *is defined, single-valued, and* (Σ, N)*-monogenic over the whole plane.*

Proof. In D, $f(z)$ must coincide with either a function of the form (8.14) or a function of the form (8.31), and has the desired properties. Since the "components" a_i of a (Σ, N)-monogenic function are analytic functions of x and y, they are uniquely determined by their values in any neighborhood.

9. (Σ, N)-monogenic functions with prescribed a_i. If $f(z) = u + iv$ is an analytic function of a complex variable $x + iy$, then u and v, the real and imaginary parts of $f(z)$, satisfy Laplace's equation. The analogue of this proposition for (Σ, N)-monogenic functions was considered in **5.** Again, given a harmonic function $u = u(x, y)$ there exist two analytic functions of $x + iy$, one having u as its real part and the other having u as its imaginary part. This section will be concerned with the analogue of this "inverse" problem for (Σ, N)- monogenic functions. (A special case has already been treated in **8.**)

To facilitate later computations, we note that

$$M^N u = \left(\frac{\partial^2}{\partial X \partial X^*} + \frac{\partial^2}{\partial Y^* \partial Y} \right)^n u = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^{2n} u}{(\partial X \partial X^*)^k (\partial Y^* \partial Y)^{n-k}},$$

$$L^N u = \left(\frac{\partial^2}{\partial X^* \partial X} + \frac{\partial^2}{\partial Y \partial Y^*} \right)^n u = \sum_{k=0}^{n} \binom{n}{k} \frac{\partial^{2n} u}{(\partial X^* \partial X)^k (\partial Y \partial Y^*)^{n-k}}. \tag{9.1}$$

We treat first the case when the real part a_0 is prescribed.

LEMMA 9.1. *Suppose that* $u = (x, y)$ *of class* $C^{(2N)}$ *satisfies the equation* $L^N u = 0$ *in a domain D. Then, there exists a function* $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ *which is* (Σ, N)*-monogenic in D and such that* $a_0 = u$.

Proof. We shall prove our result first when D is a rectangle with sides parallel to the axes. For convenience, we suppose that the origin is an interior

point of D. The proof is by induction. The conclusion is true for $N=1$ for any matrix Σ. In what follows, we suppose that the conclusion is true for $N-1$ for any matrix Σ.

It follows that given a function $v=v(x,y)$ of class $C^{(2N-2)}$ satisfying $M^{N-1}v=0$ in D, there exists a function $g(z)=\sum_{i=0}^{2N-3} b_i j_{N-1}{}^i$ which is $(\Sigma, N-1)$-monogenic in D and such that $b_1=v$. Because for the matrix Σ' of (6.1), we have

$$L'=v_{X^*X}+v_{YY^*}=Mv$$

and consequently $L'^{N-1}v=M^{N-1}v=0$ for the given function v. Hence by the induction hypothesis, there exists a function $f(z)=\sum_{i=0}^{2N-3} a_i j_{N-1}{}^i$ which is $(\Sigma', N-1)$-monogenic in D and such that $a_0=v$. From Theorem 6.1, it follows that $\sum_{i=0}^{2N-1} b_i j_{N-1}{}^i = g(z) = j_{N-1}f(z)$ is $(\Sigma, N-1)$-monogenic, and from Theorem 2.6 that $b_1=a_0=v$.

Now, let $u=u(x,y)$ of class $C^{(2N)}$ satisy $L^N u=0$ in D. Writing $a_0=u$, we seek to construct $2N-1$ functions $a_0,\cdots,a_{2N-1}$ such that $f(z)=\sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic in D. In order to do this, we shall introduce three auxiliary functions $\omega_1(x,y)$, $\omega_2(x,y)$, and $\phi(x,y)$; define $a_1(x,y)$ and $a_{2N-1}(x,y)$; and from the three functions a_0, a_1, and a_{2N-1} deduce the rest. The restriction that D be a rectangle appears in the definition of a_1 and a_{2N-1}.

Recalling that $\int\cdots dX^* = \int\cdots\sigma_2(x)dx$ and $\int\cdots dY=\int\cdots(1/\tau_1(y))dy$, we set

$$\omega_1(x,y)\equiv\int_0^y \frac{\partial a_0}{\partial X}\,dY,$$

$$\omega_2(x,y)\equiv\int_0^x \frac{\partial a_0}{\partial Y^*}\,dX^*, \tag{9.2}$$

and

$$\phi(x,y)\equiv M^{N-1}(\omega_1+\omega_2). \tag{9.3}$$

It will be shown immediately that

$$\phi(x,y)=\lambda(x)+\psi(y). \tag{9.4}$$

Since (9.2) yields

$$\frac{\partial\omega_1}{\partial Y}=\frac{\partial a_0}{\partial X}, \qquad \frac{\partial\omega_2}{\partial X^*}=\frac{\partial a_0}{\partial Y^*},$$

we may use the identities

$$\frac{\partial^{2p}}{(\partial Y^*\partial Y)^p} = \frac{\partial}{\partial Y^*}\,\frac{\partial^{2p-2}}{(\partial Y\partial Y^*)^{p-1}}\,\frac{\partial}{\partial Y},$$

$$\frac{\partial^{2p}}{(\partial X\partial X^*)^p}\,\frac{\partial}{\partial X} = \frac{\partial}{\partial X}\,\frac{\partial^{2p}}{(\partial X^*\partial X)^p},$$

to obtain from (9.2)

$$\begin{aligned} M^{N-1}\omega_1 &= \sum_{k=0}^{N-1}\binom{N-1}{k}\frac{\partial^{2N-2}\omega_1}{(\partial X\partial X^*)^k(\partial Y^*\partial Y)^{N-1-k}} \\ &= \frac{\partial^2}{\partial X\partial Y^*}\sum_{k=0}^{N-2}\binom{N-1}{k}\frac{\partial^{2N-4}a_0}{(\partial X^*\partial X)^k(\partial Y\partial Y^*)^{N-2-k}} \\ &\quad + \int_0^y \frac{\partial^{2N-1}a_0}{\partial X(\partial X^*\partial X)^{N-1}}\,dY. \end{aligned}$$

Performing a similar simplification on $M^{N-1}\omega_2$, we obtain from (9.3)

$$\begin{aligned} \phi(x,y) &= \frac{\partial^2}{\partial X\partial Y^*}\sum_{k=0}^{N-2}\binom{N-1}{k}\frac{\partial^{2N-4}a_0}{(\partial X^*\partial X)^k(\partial Y\partial Y^*)^{N-2-k}} \\ &\quad + \int_0^y \frac{\partial^{2N-1}a_0}{\partial X(\partial X^*\partial X)^{N-1}}\,dY \\ &\quad + \frac{\partial^2}{\partial X\partial Y^*}\sum_{k=1}^{N-1}\binom{N-1}{k}\frac{\partial^{2N-4}a_0}{(\partial X^*\partial X)^{k-1}(\partial Y\partial Y^*)^{N-1-k}} \\ &\quad + \int_0^x \frac{\partial^{2N-1}a_0}{\partial Y^*(\partial Y\partial Y^*)^{N-1}}\,dX^*. \end{aligned}$$

Hence

$$\begin{aligned} \phi_{X^*Y} &= \sum_{k=0}^{N-2}\binom{N-1}{k}\frac{\partial^{2N}a_0}{(\partial X^*\partial X)^{k+1}(\partial Y\partial Y^*)^{N-1-k}} \\ &\quad + \frac{\partial^{2N}a_0}{(\partial X^*\partial X)^N} \\ &\quad + \sum_{k=1}^{N-1}\binom{N-1}{k}\frac{\partial^{2N}a_0}{(\partial X^*\partial X)^k(\partial Y\partial Y^*)^{N-k}} \\ &\quad + \frac{\partial^{2N}a_0}{(\partial Y\partial Y^*)^N} \\ &= \sum_{k=1}^{N-1}\binom{N-1}{k-1}\frac{\partial^{2N}a_0}{(\partial X^*\partial X)^k(\partial Y\partial Y^*)^{N-k}} \\ &\quad + \frac{\partial^{2N}a_0}{(\partial X^*\partial X)^N} \\ &\quad + \sum_{k=1}^{N-1}\binom{N-1}{k}\frac{\partial^{2N}a_0}{(\partial X^*\partial X)^k(\partial Y\partial Y^*)^{N-k}} \end{aligned}$$

$$+\frac{\partial^{2N}a_0}{(\partial Y\partial Y^*)^N}$$

$$=\sum_{k=0}^{N}\binom{N}{k}\frac{\partial^{2N}a_0}{(\partial X^*\partial X)^k(\partial Y\partial Y^*)^{N-k}}$$

$$=L^Na_0=0.$$

by (9.2) and the hypothesis concerning a_0. Since $\phi_{X^*Y}=(\tau_1/\sigma_2)\phi_{xy}$ and $\phi_{X^*Y}=0$, we have $\phi_{xy}=0$ and (9.3) follows.

Now, let the functions $A(x)$ and $B(y)$ be defined by the equations

$$\frac{d^{2N-2}A(x)}{(dXdX^*)^{N-1}}=-\lambda(x), \tag{9.5}$$

and

$$\frac{d^{2N-2}B(y)}{(dY^*dY)^{N-1}}=-\psi(y). \tag{9.6}$$

It is easily seen that such real functions exist, although they are not uniquely determined.

It is now possible to define $a_1(x,y)$ and $a_{2N-1}(x,y)$. The system $E(\Sigma,N)$

$$\begin{aligned}&a_{2k,X}=a_{2k+1,Y},\\&a_{2k,Y^*}=a_{2k-1,X^*}-a_{2N-1,X^*}\binom{N}{k},\end{aligned} \tag{9.7}$$

$(k=0,\cdots,N-1)$ gives for $k=0$

$$\begin{aligned}&a_{0,X}=a_{1,Y},\\&a_{0,Y^*}=-a_{2N-1,X^*},\end{aligned} \tag{9.8}$$

In order to satisfy (9.8), we define [see (9.2)]

$$\begin{aligned}&a_1(x,y)\equiv\omega_1(x,y)+A(x),\\&a_{2N-1}(x,y)\equiv-\omega_2(x,y)-B(y).\end{aligned} \tag{9.9}$$

We have that

$$M^{N-1}(a_1-a_{2N-1})=0, \tag{9.10}$$

for, using (9.9)

$$\begin{aligned}M^{N-1}(a_1-a_{2N-1})&=M^{N-1}\omega_1+\frac{d^{2N-2}A(x)}{(dXdX^*)^{N-1}}\\&\quad+M^{N-1}\omega_2+\frac{d^{2N-2}B(y)}{(dY^*dY)^{N-1}}\\&=\phi-\lambda(x)-\psi(y)\\&=0,\end{aligned}$$

by (9.3) and (9.4).

Since, in view of (9.10), we have one function satisfying $M^{N-1}v = 0$, we may use equation (4.1) "backwards" to produce the missing functions $a_2, \cdots, a_{2N-2}$. By the remark made at the beginning of the proof, there exists a $(\Sigma, N-1)$-monogenic function $g(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}^i$ such that $\alpha_1 = a_1 - a_{2N-1}$. Define $a_{2N-2}(x, y)$ by the relation

$$a_{2N-2} = a_0 - \alpha_0.$$

Since a_{2N-2} and a_{2N-1} are known, we may define a_{2k} and a_{2k+1} for $k = 1, \cdots, N-2$ by the relations [see equation (4.1)]

$$\begin{aligned} a_{2k} &= \alpha_{2k} + \binom{N-1}{k} a_{2N-2}, \\ a_{2k+1} &= \alpha_{2k+1} + \binom{N-1}{k} a_{2N-1}. \end{aligned} \tag{9.11}$$

Equation (9.11) also holds for $k = 0$, and for $k = N-1$ provided α_{2N-2} and α_{2N-1} are taken as zero. We must now show that the a_i thus defined satisfy (9.7). In order to do this, we remark that the α_i satisfy $E(\Sigma, N-1)$

$$\begin{aligned} \alpha_{2k,X} &= \alpha_{2k+1,Y}, \\ \alpha_{2k,Y^*} &= \alpha_{2k-1,X^*} - \alpha_{2N-3,X^*} \binom{N-1}{k}, \end{aligned} \tag{9.12}$$

$(k = 0, \cdots, N-2)$.

First, as has already been pointed out, in view of (9.9), the pair of equations obtained by setting $k = 0$ in (9.7) are satisfied. That is

$$\begin{aligned} a_{0,X} &= a_{1,Y}, \\ a_{0,Y^*} &= -a_{2N-1,X^*}. \end{aligned} \tag{9.13}$$

On the other hand, (9.12) gives for $k = 0$

$$\begin{aligned} \alpha_{0,X} &= \alpha_{1,Y}, \\ \alpha_{0,Y^*} &= -\alpha_{2N-3,X^*}, \end{aligned} \tag{9.14}$$

and using (9.11) for $k = 0$ and $k = N-2$, (9.14) becomes

$$\begin{aligned} a_{0,X} - a_{2N-2,X} &= a_{1,Y} - a_{2N-1,Y}, \\ a_{0,Y^*} - a_{2N-2,Y^*} &= -\left[a_{2N-3,X^*} - a_{2N-1,X^*} \binom{N-1}{N-2}\right], \end{aligned} \tag{9.15}$$

which, in view of (9.13), simplifies to

$$(9.16)\qquad \begin{aligned} a_{2N-2,X} &= a_{2N-1,Y}, \\ a_{2N-2,Y^*} &= a_{2N-3,X^*} - a_{2N-1,X^*}\binom{N}{N-1}, \end{aligned}$$

the pair of equations obtained from (9.7) by setting $k = N-1$.

To prove that (9.9) holds for $k = 1, \cdots, N-2$, we first substitute into (9.12) the α_i's as given by (9.11). The result is

$$(9.17)\qquad \begin{aligned} &a_{2k,X} - \binom{N-1}{k} a_{2N-2,X} = a_{2k+1,Y} - \binom{N-1}{k} a_{2N-1,Y}, \\ &a_{2k,Y^*} - \binom{N-1}{k} a_{2N-2,Y^*} = a_{2k-1,X^*} - \binom{N-1}{k-1} a_{2N-1,X^*} \\ &\qquad - \binom{N-1}{k}\left[a_{2N-3,X^*} - \binom{N-1}{N-2} a_{2N-1,X^*}\right], \end{aligned}$$

$(k = 1, \cdots, N-2)$. In view of (9.16), (9.17) becomes (9.7). Thus $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N)-monogenic and $a_0 = u$ inside the rectangle D.

For convenience, if a domain D is such that Lemma 9.1 holds for D, then we shall call D an adequate domain. To obtain the lemma for a general *domain* D, we verify the following statements:

(1) If D_1 and D_2 are adequate domains and their intersection $D_1 \cdot D_2$ is a non-vacuous connected set, then their sum $D_1 + D_2$ is also an adequate domain.

(2) If D_1, D_2, and D_3 are adequate domains, $D_1 \cdot D_2$ and $D_2 \cdot D_3$ are non-vacuous connected sets, and D_1 and D_3 are mutually exclusive, then $D_1 + D_2 + D_3$ is an adequate domain.

(3) Let a domain be said to be of class n, $(n = 1, 2, \cdots)$, if its closure is the sum of n mutually exclusive rectangular interiors, plus their boundaries, the rectangular boundaries having their sides parallel to the x and y axes and such that if two of the boundaries have a common part, then the common part contains more than one point. For each n, every domain of class n is adequate.

(4) If a domain D is the sum of a monotone ascending sequence $D_1 \subset D_2 \subset D_3 \subset \cdots$ of adequate domains, then D is an adequate domain.

To prove (1), let $f_i(z)$, for $i = 1, 2$, be (Σ, N)-monogenic in D_i and such that $Re\, f_i(z) = u$ in D_i. $f_1(z)$ differs from $f_2(z)$ in $D_1 \cdot D_2$ by at most a null function (see Section **8**), since $Re[f_1(z) - f_2(z)] = 0$ in $D_1 \cdot D_2$.

Hence, the function $g(z) \equiv f_1(z) - f_2(z)$ is defined over the whole plane (see Section 8), and the function

$$F(z) = \begin{cases} f_1(z) \text{ in } D_1, \\ f_2(z) + g(z) \text{ in } D_2, \end{cases}$$

is (Σ, N)-monogenic in $D_1 + D_2$ and $Re[F(z)] = u$ there. (2) follows by applying (1) first to D_1 and D_2 and then to $D_1 + D_2$ and D_3. (3) is proved by induction. A domain of class 1 is the interior of a rectangle; hence, it is adequate. It is easily verified that the closure of a domain of class n is the sum of the closure of a domain of class $n-1$ and the closure of a domain of class 1. The induction is completed by introducing a suitable rectangle [to play the role of D_2 in (2)] and reasoning as in the proof of (2). (4) follows by applying repeatedly the same argument used in the proof of (1), in order to extend the definition of a (Σ, N)-monogenic function $f_1(z)$ such that $Re\, f_1(z) = u$ in D_1 to all domains of the ascending sequence.

That Lemma 9.1 holds for any domain D follows from (4), since every domain is known to be the sum of a monotone ascending sequence of domains each of which belongs to class n for some n.

We remark that if D is simply connected, then the (Σ, N)-monogenic function obtained above is single-valued, by the analogue of the monodromy theorem for (Σ, N)-monogenic functions. If D is not simply connected, then the (Σ, N)-monogenic function obtained above may be multiple valued.

From Lemma 9.1, in the same way as was pointed out for $N-1$ in the course of the proof of that lemma, we have

LEMMA 9.2. *If $v = v(x, y)$ of class $C^{(2N)}$ satisfies the equation $M^N v = 0$ in a domain D, then there exists a (Σ, N)-monogenic function $g(z) = \sum_{i=0}^{2N-1} b_i j_N^i$ such that $b_1 = v$.*

The preceding considerations enable us to give a complete solution to the "inverse problem" for (Σ, N)-monogenic functions by means of the

THEOREM 9.1. *If $u = u(x, y)$ of class $C^{(2N)}$ is such that either (1) $L^N u = 0$ or (2) $M^N u = 0$ in a domain D, then for each integer $k = 0, \cdots, N-1$ there exists (1) a function $f_k(z) = \sum_{i=0}^{2N-1} a_{ki} j_N^i$ which is (Σ, N)-monogenic in D and such that $a_{k,2k} = u$ or (2) a function $g_k(z) = \sum_{i=0}^{2N-1} b_{ki} j_N^i$ which is (Σ, N)-monogenic in D and such that $b_{k,2k+1} = u$.*

We prove only (1), the proof for (2) being analogous. Let u satisfying $L^N u = 0$ be given and let $f_0(z) = \sum_{i=0}^{2N-1} a_{0i} j_N{}^i$ be the (Σ, N)-monogenic function given by Lemma 9.1, with $a_{00} = u$. Then $f_k(z) = \sum_{i=0}^{2N-1} a_{ki} j_N{}^i = (1/K_{N,k}) f_0(z)$, $k = 0, \cdots, N-1$ are the desired functions, where the $K_{N,k}$ are given in Theorem 2.7. By Theorem 2.7, $a_{k,2k} = a_{00} = u$. By Theorem 2.8 and Corollary 6.1, it follows that the $f_k(z)$ are (Σ, N)-monogenic.

Finally, we remark that to obtain the most general (Σ, N)-monogenic function $F_k(z) = \sum_{i=0}^{2N-1} A_{ki} j_N{}^i$ such that $A_{k,2k} = u$, we merely add to the $f_k(z)$ the appropriate "null functions" from 8.

10. Formal power series and the expansion theorem. An expression of the form

$$\sum_{n=0}^{\infty} \alpha_n \cdot Z^{(n)}(z_0; z), \tag{10.1}$$

where $\alpha_n = \sum_{i=0}^{2N-1} \alpha_{ni} j_N{}^i$ are hypercomplex constants, will be called a formal power series. Our primary objective in this section is to show that if $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic at a point z_0, then it can be represented uniquely by a formal power series (10.1) in a sufficiently small neighborhood of z_0. Preliminary to this, we shall establish the existence, under certain conditions on the coefficients α_n, of a domain of convergence of (10.1) and also show that a (Σ, N)-monogenic function in a domain D is uniquely determined by the value of the function and of all its (Σ, N)- and (Σ', N)-derivatives at a point of D.

In 3 the higher (Σ, N)-derivatives $f^{[n]}(z) = \sum_{i=0}^{2N-1} a_i{}^{[n]} j_N{}^i$ of a (Σ, N)-monogenic function $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ were defined, and it was shown that $f^{[2n]}(z)$ is (Σ, N)-monogenic and $f^{[2n+1]}(z)$ is (Σ', N)-monogenic.

Equation (3.5) gives that $f^{[1]}(z) = \dfrac{d_\Sigma f}{d_\Sigma z} = \sum_{i=0}^{2N-1} a_i{}^{[1]} j_N{}^i$, where

$$\begin{aligned} a_{2k}{}^{[1]} &= a_{2k,X} = \sigma_1 a_{2k,x}, \\ a_{2k+1}{}^{[1]} &= a_{2k+1,X^*} = (1/\sigma_2) a_{2k+1,x}, \end{aligned} \tag{10.2}$$

$(k = 0, \cdots, N-1)$. Repeated application of (10.2) gives immediately

$$(10.3)\qquad a_{2k}{}^{[2n]} = \frac{\partial^{2n} a_{2k}}{(\partial X^* \partial X)^n}, \qquad a_{2k+1}{}^{[2n]} = \frac{\partial^{2n} a_{2k+1}}{(\partial X \partial X^*)^n},$$

$$a_{2k}{}^{[2n+1]} = \frac{\partial^{2n+1} a_{2k}}{\partial X (\partial X^* \partial X)^n}, \qquad a_{2k+1}{}^{[2n+1]} = \frac{\partial^{2n+1} a_{2k+1}}{\partial X^* (\partial X \partial X^*)^n},$$

$(n = 0, 1, \cdots), (k = 0, \cdots N-1)$.

We shall employ a lemma given by Bers and Gelbart (1)

LEMMA 10.1. *Let $F(x)$, $G(x)$, $H(x)$ be analytic functions of the real variable x in the neighborhood of $x = 0$. Define*

$$(10.4)\qquad F_0(x) = F(x),$$

$$F_n(x) = \begin{cases} G(x)\,\dfrac{dF_{n-1}}{dx} & \text{if } n \text{ is odd,} \\ H(x)\,\dfrac{dF_{n-1}}{dx} & \text{if } n \text{ is even.} \end{cases}$$

Then there exists a constant $C > 0$ such that

$$(10.5)\qquad |F_n(0)| < n!\, C^n,$$

for all n.

The following lemma will be used in the proof of the expansion theorem.

LEMMA 10.2. *If $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ is (Σ, N)-monogenic at $z = z_0$, then there exists a constant $C > 0$ such that*

$$(10.6)\qquad \| f^{[n]}(z_0) \| < n!\, C^n,$$

for all n.

Proof. Without loss in generality z_0 may be taken to be zero. Applying Lemma 10.1 for each $k = 0, \cdots, N-1$, with

$$F(x) = a_{2k}(x, 0), \qquad G(x) = \mathring{\sigma}_1(x), \qquad H(x) = 1/\sigma_2(x),$$

and using (10.3), we see that there exist N constants C_{2k} such that

$$(10.7)\qquad |\, a_{2k}{}^{[n]}(0, 0)\, | < n!\, C_{2k}{}^n,$$

$(k = 0, \cdots, N-1)$.

Similarly, there exist N constants C_{2k+1} such that

$$(10.8)\qquad |\, a_{2k+1}{}^{[n]}(0, 0)\, | < n!\, C_{2k+1}{}^n.$$

But, by (2.24) and (2.25)

$$\| f^{[n]}(0) \| \leqq \sum_{i=0}^{2N-1} | a_i{}^{[n]}(0,0) | \, \| j_N{}^i \| , \tag{10.9}$$

and $\| j_N{}^i \| = 1$ for $0 \leq i \leq 2N-1$, so that from (10.7) and (10.8)

$$\| f^{[n]}(0) \| < n! \, (\sum_{i=0}^{2N-1} C_i)^n. \tag{10.10}$$

THEOREM 10.1. *If* $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ *is* (Σ, N)*-monogenic at* $z = z_0$ *and*

$$f^{[n]}(z_0) = 0, \qquad n = 0, 1, \cdots, \tag{10.11}$$

then

$$f(z) \equiv 0.$$

The proof is by induction on N. For $N = 1$ the theorem has been proved by Bers and Gelbart (1). Suppose that the conclusion is true for $N-1$, and let $f(z) = \sum_{i=0}^{2N-1} a_i j_N{}^i$ be a (Σ, N)-monogenic function satisfying (10.11) at $z = z_0$. Equation (10.11) implies that

$$a_i{}^{[n]}(z_0) = 0, \qquad n = 0, 1, \cdots, \tag{10.12}$$

$(i = 0, \cdots, 2N-1)$.

By Lemma 4.1, the function $\phi(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}{}^i$, where

$$\begin{aligned} \alpha_{2k} &= a_{2k} - \binom{N-1}{k} a_{2N-2}, \\ \alpha_{2k+1} &= a_{2k+1} - \binom{N-1}{k} a_{2N-1}, \end{aligned} \tag{10.13}$$

$(k = 0, \cdots, N-2)$, is $(\Sigma, N-1)$-monogenic in a neighborhood of z_0. From (10.11), we have that

$$\phi^{[n]}(z_0) = 0, \qquad n = 0, 1, \cdots, \tag{10.14}$$

since (10.13) and (10.3) together give

$$\begin{aligned} \alpha_{2k}{}^{[n]}(z_0) &= a_{2k}{}^{[n]}(z_0) - \binom{N-1}{k} a_{2N-2}{}^{[n]}(z_0) = 0, \\ \alpha_{2k+1}{}^{[n]}(z_0) &= a_{2k+1}{}^{[n]}(z_0) - \binom{N-1}{k} a_{2N-1}{}^{[n]}(z_0) = 0, \end{aligned} \tag{10.15}$$

$(k = 0, \cdots, N-2)$, using (10.12).

By (10.14) and the induction hypothesis, we have that $\phi(z) = \sum_{i=0}^{2N-3} \alpha_i j_{N-1}^i \equiv 0$, and from (10.13)

$$(10.16) \qquad \begin{aligned} a_{2k} &= \binom{N-1}{k} a_{2N-2}, \\ a_{2k+1} &= \binom{N-1}{k} a_{2N-1}, \end{aligned}$$

$(k = 0, \cdots, N-2)$. Since $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ is (Σ, N)-monogenic, then

$$(10.17) \qquad \begin{aligned} a_{2k,X} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= a_{2k-1,X^*} - a_{2N-1,X^*} \binom{N}{k}, \end{aligned}$$

$(k = 0, \cdots, N-1)$. Replacing a_{2k} and a_{2k+1} in (10.17)' by their values from (10.16) yields

$$(10.18) \qquad \begin{aligned} a_{2N-2,X} &= a_{2N-1,Y}, \\ a_{2N-2,Y^*} &= -a_{2N-1,X^*}. \end{aligned}$$

Replacing a_{2N-2} and a_{2N-1} in (10.18) by their values from (10.16) gives

$$(10.19) \qquad \begin{aligned} a_{2k,X} &= a_{2k+1,Y}, \\ a_{2k,Y^*} &= -a_{2k+1,X^*}, \end{aligned}$$

$(k = 0, \cdots, N-1)$. This last equation shows that the N functions

$$(10.20) \qquad f_k(z) = a_{2k} + j_1 a_{2k+1},$$

$(k = 0, \cdots, N-1)$, are $(\Sigma, 1)$-monogenic. But from (10.12) and (10.3), we have that

$$(10.21) \qquad f_k^{[n]}(z_0) = 0, \qquad n = 0, 1, \cdots$$

$(k = 0, \cdots, N-1)$. Hence, $a_i = 0$ for $i = 0, \cdots, 2N-1$ and $f(z) \equiv 0$.

Returning to the consideration of the formal power series (10.1), we remark that such a series represents a (Σ, N)-monogenic function in any domain in which it converges uniformly, and may be (Σ, N)-integrated term by term in any such domain. We shall show that a domain of uniform convergence of (10.1) is determined by the sequence of coefficients α_n.

THEOREM 10.2. *Let* $\alpha_n = \sum_{i=0}^{2N-1} \alpha_{ni} j_N^i$ *be a sequence of numbers of* A_{2N} *and* $z = z_0$ *be fixed. If*

$$(10.22) \qquad \frac{1}{\overline{\lim} \sqrt[n]{\| \alpha_n \|}} = R,$$

does not vanish, then there exists a neighborhood D of z_0 such that the formal power series

$$f(z) = \sum_{n=0}^{\infty} \alpha_n \cdot Z^{(n)}(z_0; z), \tag{10.23}$$

and all the formal power series obtained by successively (Σ, N)- and (Σ', N)-differentiating (10.23) *term by term converge uniformly and absolutely in D.*

Proof. Let D be the domain

$$|z - z_0| < \frac{\rho}{2^{\frac{1}{2}} \Gamma_N m_\Sigma (|z_0| + R)}, \qquad 0 < \rho < R. \tag{10.24}$$

From (10.22), it follows that D has the required property, for the inequality (7.14) implies

$$\sum_{n=0}^{M} \| \alpha_n \cdot Z^{(n)}(z_0; z) \| < 2 \Gamma_N \sum_{n=0}^{M} \| \alpha_n \| \rho^n,$$

and similar inequalities hold for the derived series.

Successive termwise (Σ, N)- and (Σ', N)-differentiation of (10.23) and evaluation at z_0 of the resulting formal power series gives the "Taylor formulas"

$$f^{[n]}(z_0) = n! \, \alpha_n, \tag{10.25}$$

$n = 0, 1, \cdots$.

THEOREM 10.3. *Let $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ be (Σ, N)-monogenic at $z = z_0$. Then $f(z)$ can be (uniquely) expanded at $z = z_0$ in a formal power series of the form* (10.23).

Proof. Let the sequence of hypercomplex numbers α_n be defined by (10.25). By virtue of Lemma 10.2

$$\overline{\lim} \sqrt[n]{\| \alpha_n \|} \leq 1'.$$

Hence, by Theorem 10.2, the formal power series

$$g(z) = \sum_{n=0}^{\infty} \alpha_n \cdot Z^{(n)}(z_0; z),$$

converges uniformly and absolutely in a neighborhood of z_0 and

$$g^{[n]}(z_0) = n! \, \alpha_n,$$

$n = 0, 1, \cdots$. Then $f(z) - g(z) \equiv 0$ by Theorem 10.1.

Theorems 10.1 and 10.3 serve to show that the class of functions $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ which can be expanded in a formal power series of the form (10.1) in a neighborhood of z_0 coincides with the class of functions which are (Σ, N)-monogenic at z_0 according to Definition 3.1. This corresponds to the equivalence of the Cauchy and Weierstrass definitions of an analytic function.

11. Complete systems of particular solutions of $L^N u = 0$. We define

$$U_{Nnv}(x_0, y_0; x, y) = Re[j_N^v \cdot Z^{(n)}(z_0; z)], \tag{11.1}$$

where $z = x + iy$, $z_0 = x_0 + iy_0$, $n = 0, 1, \cdots, v = 0, 1, \cdots, 2N-1$. Obviously, each U is a (real-valued) particular solution of the equation $L^N u = 0$, analytic in the whole plane. Notice that for $n < N$ some of the functions (11.1) vanish. The sequence $\{U_{Nnv}\}$ has the completeness properties mentioned in Section 1. We have

THEOREM 11.1. *Each solution $u = u(x, y)$ of the equation $L^N u = 0$, which is of class $C^{(2N)}$ in a domain containing the point z_0, admits the expansion*

$$u(x, y) = \sum_{n=0}^{\infty} \sum_{v=0}^{2N-1} a_{nv} U_{Nnv}(x_0, y_0; x, y), \tag{11.2}$$

where the a's are real constants and the series converges uniformly and absolutely in some neighborhood of z_0.

Proof. By Lemma 9.1, there exists, in a suitable neighborhood of z_0, a (Σ, N)-monogenic function $f(z) = \sum_{i=0}^{2N-1} a_i j_N^i$ such that $a_0 = u$. By Theorem 10.3, we have

$$f(z) = \sum_{n=0}^{\infty} \mathfrak{a}_n \cdot Z^{(n)}(z_0; z), \tag{11.3}$$

where

$$\mathfrak{a}_n = \sum_{v=0}^{2N-1} a_{nv} j_N^v,$$

$n = 0, 1, \cdots$, in a sufficiently small neighborhood of z_0. (11.2) follows upon equating the real parts of both sides of (11.3), using (7.8) and (11.1).

The sequence of real functions (11.1) depends upon the point z_0. In order to prove a more general expansion theorem, we establish

THEOREM 11.2. *Let z_0 and b be points of the plane and let α be a number of A_{2N}, then*

$$\alpha \cdot Z^{(n)}(z_0; z) = \sum_{k=0}^{n} \binom{n}{k} A_k \cdot Z^{(n-k)}(b; z), \tag{11.4}$$

where

$$A_k = \begin{cases} \alpha \cdot Z^{(k)}(z_0; b) & \text{if } k \text{ is even,} \\ \alpha \cdot \bar{Z}^{(k)}(z_0; b) & \text{if } k \text{ is odd.} \end{cases} \tag{11.5}$$

Equation 11.4 is another analogue of the binomial formula. This theorem follows immediately from the expansion theorem 10.3, with the aid of (7.3) and (10.25). We note that since (11.4) is a relation between iterated integrals of the functions σ_i and τ_i, the equation holds independently of the expansion theorem and of the analyticity of the σ_i and τ_i.

We now define

$$U_{Nn\nu}(x, y) = U_{Nn\nu}(0, 0; x, y). \tag{11.6}$$

From Theorem 11.2, taking the origin as b, we obtain from (11.1) and (11.6)

$$U_{Nn\nu}(x_0, y_0; x, y) = \sum_{m=0}^{n} \sum_{\mu=0}^{2N-1} b_{Nn\nu m\mu} U_{Nm\mu}(x, y), \tag{11.7}$$

where $n = 0, 1, \cdots$; $\nu = 0, 1, \cdots, 2N-1$, and the b's are real constants. Hence, we have the

THEOREM 11.3. *Each solution $u = u(x, y)$ of the equation $L^N u = 0$, which is of class $C^{(2n)}$ in a domain containing the point z_0, admits the expansion*

$$u(x, y) = \sum_{n=0}^{\infty} \{ \sum_{\nu=0}^{2N-1} \sum_{m=0}^{n} \sum_{\mu=0}^{2N-1} c_{n\nu m\mu} U_{Nm\mu}(x, y) \}, \tag{11.8}$$

where the c's are real constants and the series converges uniformly and absolutely in some neighborhood of z_0.

Finally, we observe that by (7.4), (7.5), and (7.10) to (7.13), the $U_{Nn\nu}$ are linear combinations with real constant coefficients of the functions given in (7.9), which are independent of N.

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ON HIGHER NORMAL SPACES FOR V_m IN S_n.*

By Abraham Schwartz

C. Segre has proved that a real V_m in S_n whose points are all axial points is either contained in a geodesic V_{m+1} or has a second fundamental tensor of rank 1.[1] N. Coburn has studied the case of a real V_m in S_n whose points are all planar.[2] C. B. Allendoerfer has pointed out that if an open, simply connected domain of a Riemannian manifold imbedded within a Euclidean manifold is of "type" ≥ 3 at every point, then the second and higher normal spaces vanish.[3]

In this paper we consider a V_m in an S_n, and let the first normal space be of k dimensions at a particular point P. We prove first

THEOREM 1. *If the second normal space is of j dimensions, then*

$$j \leq \tfrac{1}{6}t(t+1)(t+2) + \tfrac{1}{2}u(u+1),$$

where t and u are integers determined by k in accordance with the following inequalities:

$$\tfrac{1}{2}t(t+1) \leq k < \tfrac{1}{2}(t+1)(t+2), \qquad u = k - \tfrac{1}{2}t(t+1).$$

Then we prove

THEOREM 2. *Either*

1) *The second normal space at the point P vanishes, or;*

2) *The three index curvature tensor, $\overset{1}{H}_{\alpha\nu}{}^{\lambda}$, is of rank $\leq k$ with respect to the index α at P, or;*

3) *The V_m can be imbedded in a V_{m+r}, $0 < r < k$, in such a way that*

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* Received April 22, 1946.

[1] Schouten and Struik, *Einfuhrung in die Neueren Methoden der Differentialgeometrie*, vol. 2, Groningen, p. 99.

[2] N. Coburn, "V_m in S_n with planar points ($m \geq 3$)" *Bulletin of the American Mathematical Society*, vol. 45 (1939), pp. 774-783.

[3] C. B. Allendoerfer, "Rigidity for spaces of class greater than one," *American Journal of Mathematics*, vol. 61 (1939), p. 636.

1° *$\overset{1}{H}_{ab}{}^{\lambda}$, the three index curvature tensor for the V_m in the V_{m+r}, is of rank $\leq r$ with respect to the index a at P.*[4]

Note that if the first normal space is everywhere of k dimensions and that if the second normal space vanishes everywhere, then the V_m can be imbedded in an S_{m+k}.

1. If k and j are the dimensionalities of the first and second normal spaces, one of the Ricci equations for a V_m in an S_n can be written in the form

$$\overset{1}{H}_{ab}{}^{\nu}\overset{2}{H}_{cv}{}^{\lambda} = \overset{1}{H}_{cb}{}^{\nu}\overset{2}{H}_{av}{}^{\lambda}, \qquad a, b, c = 1, \cdots, m; \nu, \lambda = 1, \cdots, n, \tag{1.1}$$

where the rank of $\overset{1}{H}_{ab}{}^{\nu}$ with respect to the index ν is k, and the rank of $\overset{2}{H}_{cv}{}^{\lambda}$ with respect to the index λ is j.[5] Since $\overset{1}{H}_{ab}{}^{\nu}$ is symmetric with respect to its first two indices, (1.1) says that the tensor

$$J_{abc}{}^{\lambda} = \overset{1}{H}_{ab}{}^{\nu}\overset{2}{H}_{cv}{}^{\lambda}, \tag{1.2}$$

which is of rank j with respect to the index λ, is symmetric with respect to its first three indices. Because of this symmetry we get the bound

$$j \leq \tfrac{1}{6}m(m+1)(m+2) \tag{1.3}$$

immediately by considering the number of combinations of m things taken 3 at a time, repetitions allowed,[6] but this bound can be improved considerably.

Whenever

$$\overset{1}{H}_{ab}{}^{\nu} = x\overset{1}{H}_{cd}{}^{\nu} + y\overset{1}{H}_{ef}{}^{\nu} + \cdots + z\overset{1}{H}_{ij}{}^{\nu}, \tag{1.4a}$$

we have

$$J_{abg}{}^{\lambda} = \overset{1}{H}_{ab}{}^{\nu}\overset{2}{H}_{gv}{}^{\lambda} = xJ_{cdg}{}^{\lambda} + yJ_{efg}{}^{\lambda} + \cdots + zJ_{ijg}{}^{\lambda}. \tag{1.4b}$$

One can select a set of k independent vectors of the first normal space from the set $\overset{1}{H}_{ab}{}^{\lambda}$, $a, b = 1, \cdots, m$. Let α be the number of these vectors which have 1 as one of their indices. By interchanging indices if necessary, we can be sure that no greater number of these vectors have some other integer as

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[4] For the definition of the three index curvature tensor see Schouten and Struik, vol. 2, p. 80. For the definition of rank with respect to a particular index, see Schouten and Struik, vol. 1, p. 19.

[5] See Schouten and Struik, vol. 2, pp. 119, 122, and 124.

[6] A better bound than that given in Schouten and Struik, vol. 2, p. 116.

one of their indices. Let β be the number of the remaining k-α vectors which have 2 as one of their indices. Again we can assume that no greater number of the remaining vectors have some other integer as one of their indices. We repeat this process until the k vectors have been exhausted, and then we can list our k independent vectors as

$$\begin{array}{c} \overset{1}{H}_{1a_1}{}^{\lambda}, \overset{1}{H}_{1a_2}{}^{\lambda}, \cdots, \overset{1}{H}_{1a_\alpha}{}^{\lambda} \\ \overset{1}{H}_{2b_1}{}^{\lambda}, \overset{1}{H}_{2b_2}{}^{\lambda}, \cdots, \overset{1}{H}_{2b_\beta}{}^{\lambda} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ \overset{1}{H}_{sn_1}{}^{\lambda}, \overset{1}{H}_{sn_2}{}^{\lambda}, \cdots, \overset{1}{H}_{sn_\nu}{}^{\lambda}, \end{array} \tag{1.5}$$

where

$$\alpha \geq \beta \geq \gamma \geq \cdots \geq \nu, \qquad \alpha + \beta + \gamma + \cdots + \nu = k. \tag{1.6}$$

Consider the vectors $J_{ab_1}{}^{\lambda}$ of the second normal space. Because $\overset{1}{H}_{ab}{}^{\lambda}$ is a linear combination of the vectors in list (1.5), the vectors $J_{ab_1}{}^{\lambda}$ are linear combinations of the vectors

$$J_{1a_{11}}{}^{\lambda}, J_{1a_{21}}{}^{\lambda}, \cdots, J_{1a_{\alpha 1}}{}^{\lambda}, J_{2b_{11}}{}^{\lambda}, \cdots, J_{2b_{\beta 1}}{}^{\lambda}, \cdots, J_{sn_{11}}{}^{\lambda}, \cdots, J_{sn_{\nu 1}}{}^{\lambda}. \tag{1.7}$$

Consider next the vectors $J_{ab_2}{}^{\lambda}$. They are linear combinations of the vectors

$$J_{1a_{12}}{}^{\lambda}, J_{1a_{22}}{}^{\lambda}, \cdots, J_{1a_{\alpha 2}}{}^{\lambda} \tag{1.8}$$

and

$$J_{2b_{12}}{}^{\lambda}, J_{2b_{22}}{}^{\lambda}, \cdots, J_{2b_{\beta 2}}{}^{\lambda}, J_{3c_{12}}{}^{\lambda}, \cdots, J_{sn_{12}}{}^{\lambda}, \cdots, J_{sn_{\nu 2}}{}^{\lambda}. \tag{1.9}$$

Because $J_{1ac}{}^{\lambda} = J_{ac_1}{}^{\lambda}$ the vectors of list (1.8) are dependent on those of list (1.7), but those of list (1.9) are perhaps independent. Similarly, when we consider the vectors $J_{ab_3}{}^{\lambda}$ we will introduce the new vectors

$$J_{3c_{13}}{}^{\lambda}, \cdots, J_{3c_{\gamma 3}}{}^{\lambda}, \cdots, J_{sn_{13}}{}^{\lambda}, \cdots, J_{sn_{\nu 3}}{}^{\lambda}, \tag{1.10}$$

and finally, when we consider the vectors $J_{abs}{}^{\lambda}$ we will introduce the new vectors

$$J_{sn_{1s}}{}^{\lambda}, J_{sn_{2s}}{}^{\lambda}, \cdots, J_{sn_{\nu s}}{}^{\lambda}. \tag{1.11}$$

But any vector $J_{abc}{}^{\lambda}$ is linearly dependent on the vectors $J_{1a_1c}{}^{\lambda}, J_{1a_2c}{}^{\lambda}, \cdots, J_{1a_\alpha c}{}^{\lambda}, J_{2b_1c}{}^{\lambda}, \cdots, J_{sn_{\nu c}}{}^{\lambda}$, which are in turn dependent on the vectors of lists (1.7), (1.9), (1.10), (1.11) because of the J tensor symmetry. Thus, counting the vectors in (1.7), (1.9), (1.10), (1.11), we have

$$j \leq 1\alpha + 2\beta + 3\gamma + \cdots + s\nu. \tag{1.12}$$

We shall show below that a coordinate system can always be found in which we shall have not only (1.6), but

$$\alpha > \beta > \gamma > \cdots > \nu. \tag{1.13}$$

Theorem 1 now follows from (1.12) and (1.13). For, to make the right hand side of (1.12) a maximum, one must choose

$$\alpha = t+1,\ \beta = t,\ \gamma = t-1, \cdots, \epsilon = t+2-u,\ \zeta = t+1-(u+1), \\ \eta = t+1-(u+2), \cdots, \nu = 1, \tag{1.14}$$

where the integers t and u have been determined by k in the following way:

$$\tfrac{1}{2}t(t+1) \leq k < \tfrac{1}{2}(t+1)(t+2), \qquad u = k - \tfrac{1}{2}t(t+1). \tag{1.15}$$

The maximum value of the right hand side of (1.12) is then

$$\sum_{n=1}^{u} n(t+2-n) + \sum_{n=u+1}^{t} n(t+1-n) \\ = \tfrac{1}{6}t(t+1)(t+2) + \tfrac{1}{2}u(u+1). \tag{1.16}$$

2. To complete the proof of Theorem 1 it is necessary to demonstrate (1.13).

LEMMA 1. *We can always assume that the index a_1 in the set* (1.5) *is* 1.

The lemma is obviously true if $\overset{1}{H}_{11}{}^{\lambda} \neq 0$. If $\overset{1}{H}_{11}{}^{\lambda} = 0$, we can assume that $\overset{1}{H}_{z1}{}^{\lambda} \neq 0$ and choose x so that the coordinate transformation

$$A_{a'}{}^{a} = \begin{cases} 1 \text{ if } a' = a \\ x \text{ if } a' = 1',\ a = z \\ 0 \text{ otherwise} \end{cases} \tag{2.1}$$

will carry us into a new coordinate system in which the lemma is satisfied.

LEMMA 2. *We can assume that the indices $b_1, b_2, \cdots, b_\beta$ in the set* (1.5) *are included within the indices $a_1, \cdots, a_\alpha$.*

For, suppose that $b_1, b_2, \cdots, b_i$ are not included in the set $a_1, a_2, \cdots, a_\alpha$. Then the coordinate transformation

$$A_{a'}{}^{a} = \begin{cases} 1 \text{ if } a' = a \\ x \text{ if } a' = 1',\ a = 2 \\ 0 \text{ otherwise} \end{cases} \tag{2.2}$$

will leave unchanged any vector $\overset{1}{H}_{ab}{}^{\lambda}$ for which neither a nor b is 1, and will replace $\overset{1}{H}_{11}{}^{\lambda}$ by $\overset{1}{H}_{1'1'}{}^{\lambda} = \overset{1}{H}_{11}{}^{\lambda} + 2x\overset{1}{H}_{12}{}^{\lambda} + x^2\overset{1}{H}_{22}{}^{\lambda}$, $\overset{1}{H}_{a1}{}^{\lambda}$ by $\overset{1}{H}_{a'1'}{}^{\lambda} = \overset{1}{H}_{a1}{}^{\lambda} + x\overset{1}{H}_{a2}{}^{\lambda}$, $a \neq 1$. The number x can be chosen so that there are $\alpha + i$ independent vectors in the set $\overset{1}{H}_{a_1'1'}{}^{\lambda}$, $\overset{1}{H}_{a_2'1'}{}^{\lambda}, \cdots, \overset{1}{H}_{a_\alpha'1'}{}^{\lambda}$, $\overset{1}{H}_{b_1'1'}{}^{\lambda}, \cdots, \overset{1}{H}_{b_i'1'}{}^{\lambda}$, and Lemma 2 holds in the new coordinate system.

These lemmas, together with the fact that $\overset{1}{H}_{12}{}^{\lambda} = \overset{1}{H}_{21}{}^{\lambda}$, make it possible to assert that $\alpha > \beta$. By proceeding in a similar fashion we can show that all of the inequalities of (1.13) hold.

3. The Ricci equation (1.1) can be rewritten in the form

$$\overset{p}{h}_{ab}\overset{Q}{v}_c = \overset{d}{h}_{cb}\overset{Q}{v}_a, \qquad p = 1, \cdots, k;\ Q = 1, \cdots, j;\ a, b, c = 1, \cdots, m, \tag{3.1}$$

the vanishing of the $\overset{Q}{v}_c$ being a necessary and sufficient condition that the second and higher normal spaces vanish. Let the index Q be fixed and write $\overset{Q}{v}_c \equiv v_c$. If the rank of the matrix

$$\left\| \begin{matrix} v_1 & v_1 & \cdot & \cdot & \cdot & v_1 \\ 1 & 2 & & & & k \\ v_2 & v_2 & \cdot & \cdot & \cdot & v_2 \\ 1 & 2 & & & & k \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ v_m & v_m & \cdot & \cdot & \cdot & v_m \\ 1 & 2 & & & & k \end{matrix} \right\| \tag{3.2}$$

is 0 no matter what index Q has been chosen, then $j = 0$ and the second normal space vanishes. If for some choice of Q the rank of (3.2) is r, $0 < r \leq k$, then one can find $m - r$ linearly independent vectors $\underset{i}{\lambda^c}$, $i = 1, \cdots, m - r$, with the property

$$\underset{i}{\lambda^c} v_c = 0, \qquad i = 1, \cdots, m - r;\ p = 1, \cdots, k, \tag{3.3}$$

and it would follow from (3.1) that

$$\underset{i}{\lambda^c} v_a \overset{p}{h}_{cb} = 0. \tag{3.4}$$

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[7] Apply equations (13.47) and (13.45), p. 126, Schouten and Struik, vol. 2, to (1.1) of this paper. Equations (13.53c), p. 128, Schouten and Struik, vol. 2 are

But we can always replace the second fundamental tensors $\overset{p}{h}_{ab}$ by a new set $\overset{p}{l}_{ab}$ derived from the old by using an orthogonal transformation on the index p. Since the matrix (3.2) is of rank r, it is possible to choose the index a in (3.4) in r essentially different ways and to find a new set of second fundamental forms, $\overset{p}{l}_{ab}$, such that

$$\sum_i \lambda^a \overset{a}{l}_{cb} = 0, \qquad \alpha = 1, \cdots, r; \; i = 1, \cdots, m - r. \tag{3.5}$$

If $r = k$, (3.5) becomes

$$\sum_i \lambda^a \overset{p}{l}_{ab} = 0, \qquad p = 1, \cdots, k; \; i = 1, \cdots, m - k. \tag{3.6}$$

Since

$$\overset{1}{H}_{ab^\nu} = \sum_p \overset{p}{l}_{ab} i^\nu, \tag{3.7}$$

we have

$$\sum_i \lambda^a \overset{1}{H}_{ab^\nu} = 0, \qquad i = 1, \cdots, m - k. \tag{3.8}$$

Hence, in this case, the rank of $\overset{1}{H}_{ab^\nu}$ with respect to the index a is at most k.

If $r < k$, the local R_m and the r unit vectors i^ν of the first normal space which correspond to the r second fundamental forms $\overset{a}{l}_{ab}$ mentioned in (3.5) will determine an $(m + r)$-dimensional direction at the point P. A V_{m+r} which contains the V_m and this $(m + r)$-dimensional direction at the point P can then be found, and if we let $\overset{1^*}{H}_{ab^\nu}$ be the three index curvature tensor of the V_m with respect to this V_{m+r}, we can write

$$\overset{1^*}{H}_{ab^\nu} = \sum_a \overset{a}{l}_{ab} i^\nu.^8 \tag{3.9}$$

(3.5) can then be written as

$$\sum_i \lambda^a \overset{1^*}{H}_{ab^\nu} = 0, \qquad i = 1, \cdots, m - r, \tag{3.10}$$

and hence the rank of $\overset{1^*}{H}_{ab^\nu}$ with respect to the index a is at most r in this case. Theorem 2 has now been demonstrated.

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correct only for the cases $y \geq 3$. Equation (3.1) of this paper is the correct statement for the case $y = 2$ if the V_n is an S_n.

[8] See Schouten and Struik, vol. 2, p. 91.

4. Allendoerfer has shown that the second normal space vanishes at a particular point if the "type" there is ≥ 3. When the type is ≥ 3, one can find a coordinate system in which the matrix

$$(4.1) \qquad \left\| \begin{matrix} \overset{1}{l}_{11} & \overset{1}{l}_{21} & \overset{1}{l}_{31} & \overset{2}{l}_{11} & \overset{2}{l}_{21} & \overset{2}{l}_{31} & \cdots & \overset{k}{l}_{11} & \overset{k}{l}_{21} & \overset{k}{l}_{31} \\ \overset{1}{l}_{12} & \overset{1}{l}_{22} & \overset{1}{l}_{32} & \overset{2}{l}_{12} & \overset{2}{l}_{22} & \overset{2}{l}_{32} & \cdots & \overset{k}{l}_{12} & \overset{k}{l}_{22} & \overset{k}{l}_{32} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \overset{1}{l}_{1m} & \overset{1}{l}_{2m} & \overset{1}{l}_{3m} & \overset{2}{l}_{1m} & \overset{2}{l}_{2m} & \overset{2}{l}_{3m} & \cdots & \overset{k}{l}_{1m} & \overset{k}{l}_{2m} & \overset{k}{l}_{3m} \end{matrix} \right\|$$

is of rank $3k$.[9] It would not now be possible to find a set of vectors $\underset{i}{\lambda^c}$, $i = 1, \cdots, m - r$, $r \leq k$ which satisfies (3.5), for if such vectors existed, (3.5) would say directly that the matrix

$$(4.2) \qquad \left\| \begin{matrix} \overset{1}{l}_{11} & \overset{1}{l}_{21} & \overset{1}{l}_{31} & \overset{2}{l}_{11} & \overset{2}{l}_{21} & \overset{2}{l}_{31} & \cdots & \overset{r}{l}_{11} & \overset{r}{l}_{21} & \overset{r}{l}_{31} \\ \overset{1}{l}_{12} & \overset{1}{l}_{22} & \overset{1}{l}_{32} & \overset{2}{l}_{12} & \overset{2}{l}_{22} & \overset{2}{l}_{32} & \cdots & \overset{r}{l}_{12} & \overset{r}{l}_{22} & \overset{r}{l}_{32} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \overset{1}{l}_{1m} & \overset{1}{l}_{2m} & \overset{1}{l}_{3m} & \overset{2}{l}_{1m} & \overset{2}{l}_{2m} & \overset{2}{l}_{3m} & \cdots & \overset{r}{l}_{1m} & \overset{r}{l}_{2m} & \overset{r}{l}_{3m} \end{matrix} \right\|$$

was of rank $\leq r$, and hence that (4.1) was of rank $< 3k$. Thus, whenever Allendoerfer's theorem predicts the vanishing of the second normal space, so does Theorem 2.

On the other hand, there are cases where Theorem 2 predicts the vanishing of the second normal space at a particular point while Allendoerfer's theorem does not. The simplest such cases arise when $k = 2$, $m = 3$. In these cases, the "type" would be ≤ 1, and hence Allendoerfer's Theorem would make no prediction. But one can easily find two three-rowed symmetric matrices for which vectors $\underset{i}{\lambda^c}$, $i = 1, \cdots, 3 - r$, $r \leq 2$, which satisfy (3.5) cannot be found, and in these cases Theorem 2 would predict the vanishing of the second normal space.

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[9] C. B. Allendoerfer, "Rigidity for spaces of class greater than one," *American Journal of Mathematics*, vol. 61 (1939), pp. 635 and 636.

CONCERNING CONTINUA IRREDUCIBLE ABOUT n POINTS.*

By R. H. Sorgenfrey.

It is the purpose of this paper to characterize, in terms of combining properties of subcontinua, continua in metric spaces which are irreducible about n points.[1] R. L. Moore has proved [2] that if a plane atriodic continuum contains no subcontinuum which separates the plane, then it is irreducible about some two points. The present author later generalized this result by showing [3] that if a unicoherent continuum fails to be a triod, then it is irreducible about some two points. The conditions of the hypothesis of this theorem, although sufficient for irreducibility, are not necessary, as simple examples exist of non-unicoherent continua which are irreducible. In this paper a condition is given which is both necessary and sufficient for irreducibility about n points.

A finite number of subcontinua of a continuum M will be said to form a *proper decomposition* of M provided that their sum is M and that no one of them is a subset of the sum of the others.

Theorem. *In order that the continuum M be irreducible about some n points, $n \geqq 2$, it is necessary and sufficient that for every proper decomposition of M into $n+1$ continua, the sum of some n of these fails to be connected.*

The necessity of the condition is obvious. The proof of its sufficiency will be shortened if the condition is formulated as follows: The continuum M will be said to have *property* B_n provided that for every decomposition of M into $n+1$ continua, the sum of some n of these fails to be connected. A proper decomposition of M into $n+1$ continua, the sum of each n of which is connected, will be called a B_n *contradicting decomposition* of M (abbre-

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[1] The term "irreducible" as here used pertains to subcontinua: A continuum M is irreducible about the n points $P_1, P_2, \cdots, P_n$ provided that the only subcontinuum of M containing all these points is M itself. A characterization of connected spaces irreducible about n points relative to connected subsets was obtained in 1931 by E. Čech ("Sur les ensembles connexes irreducibles entre n points," *Casopis*, vol. 61, pp. 109-129).

[2] "Concerning compact continua which contain no subcontinuum which separates the plane," *Proceedings of the National Academy of Sciences*, vol. 20 (1934), pp. 41-45.

[3] "Concerning triodic continua," *American Journal of Mathematics*, vol. 66 (1944), pp. 439-460.

viated B_n c. d.). Thus a B_1 c. d. is simply a decomposition of M into two proper subcontinua. The proof of the sufficiency will be by induction. Two lemmas will first be established.

LEMMA 1. *Suppose that the continuum M has property B_n $(n \geqq 2)$ and that the continua $H_1, H_2, \cdots, H_n$ form a B_{n-1} c. d. of M. Denote by S_r the sum of all the continua $H_1, H_2 \cdots, H_n$ except H_r, and let $D_r = M - S_r$. Then* (a) *D_r is connected,* (b) *there does not exist a decomposition of $\bar{D}_r$ into two proper subcontinua each of which intersects S_r, and* (c) *if D_r is indecomposable, not every composant*[4] *of $\bar{D}_r$ intersects S_r.*

Proof. The set $\bar{D}_r$ cannot have infinitely many components for if it did it would be possible to construct a B_n c. d. of M into continua[5] of the form $S_r + U_j$ $(j = 1, \cdots, n+1)$ where the U_j's are mutually separated sets whose sum is D_r and whose components are components of D_r. Hence each component of D_r is open in M. Suppose that D_r has more than one component and denote the closures of two of them by E and F. Let R be the sum of all components of $H_r - (E + F)$ which have limit points in E and let T be the sum of all those which have limit points in F. Then $E + \bar{R}$ and $F + \bar{T}$ are distinct continua whose sum is H_r. But these two continua, together with all the continua $H_1, \cdots, H_n$, except H_r, form a B_n c. d. of M. This establishes (a).

To prove (b) suppose that $\bar{D}_r$ is the sum of the proper subcontinua E and F, each of which intersects S_r. Since neither E nor F contains D_r, each of them contains an open subset of D_r not intersecting the other. Hence an argument identical with the one given in the paragraph above will lead to a contradiction.

To prove (c) suppose that $\bar{D}_r$ is indecomposable and that each of its composants intersects S_r. Let G be the upper semi-continuous collection whose elements are S_r and the individual points of D_r, let M' be the hyperspace of the decomposition G, and let f be the associated transformation. Then f is continuous and monotone. The continuum M' is not indecomposable since each two of its points lie on a proper subcontinuum of M'. It is therefore the sum of two proper subcontinua E' and F'. Let $f^{-1}(E') = E$ and $f^{-1}(F') = F$. Then E and F are proper subcontinua of M each of which

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[4] By a *composant* of an indecomposable continuum is meant a subset which is maximal with respect to the properties of being continuumwise connected and proper.

[5] The fact that $S_r + U_j$ is a continuum follows from a modification of a theorem of W. A. Wilson: *If K is a closed subset of the continuum M, then every component of $M - K$ has a limit point in K.* This theorem will be used repeatedly in the proof.

contains S_r. Since $M - E$ can have only a finite number of components, each of them is an open subset of $\bar{D}_r$. Let U be one of these components. Then $\bar{U}$ is a proper subcontinuum of $\bar{D}_r$ which contains an open subset of $\bar{D}_r$ which, since $\bar{D}_r$ is indecomposable, is impossible.

LEMMA 2. *If M has property B_2 and H and K are subcontinua of M such that $M - H$ and $M - K$ are connected and $H \cdot (M - K)$ and $K \cdot (M - H)$ contain open subsets of M, then any subcontinuum L of M which intersects both H and K contains $M - (H + K)$.*

Proof. Suppose that the lemma is not true. The set $U = M - (H + K + L)$ is connected by Lemma 1(a) (for $n = 2$, taking $H_1 = H + K + L$ and $H_2 = \overline{M - H}$). The continuum $\bar{U}$ cannot intersect both $H + L$ and $K + L$ for, if it did, these three continua would form a B_2 c. d. of M. Hence $\bar{U}$ fails to intersect $H + L$, say. But then $M - K$, which intersects both U and $H + L$ and is a subset of their sum, would fail to be connected, contrary to hypothesis.

Proof of the sufficiency of the condition. The proof will first be given for $n = 2$, that is, it will be shown that if the continuum M has property B_2, it is irreducible between some two points. Since every indecomposable continuum is irreducible about some two points, it will be assumed that M is the sum of two proper subcontinua H and K. Let $M - K = U$ and $M - H = V$. It will now be shown that there exists a point X of U such that any subcontinuum of M containing X and a point of K contains U. Suppose that no such point exists.

By Lemma 1(a) U is connected. Denote it by U_0 and $\bar{U} \cdot K$ by N_0. There exists a countable set $C = P_1 + P_2 + \cdots$ of U_0 such that $\bar{C} = \bar{U}_0$. Denote P_1 by X_1. There exists a subcontinuum H_1 of $\bar{U}_0$ irreducible from X_1 to N_0. Let $N_1 = N_0 + H_1$. If $U_0 - N_1$ is non-vacuous, it is, by Lemma 1(a), connected; denote it by U_1. Let n_2 be the smallest integer n such that P_n belongs to U_1, and denote P_{n_2} by X_2. There exists a subcontinuum H_2 of $\bar{U}_1$ irreducible from X_2 to N_1. Let $N_2 = N_1 + H_2$. This process may be continued unless for some integer r, $U_0 - N_r$ is vacuous. Suppose this to be the case and denote X_r by X. Let L be a subcontinuum of M containing X and a point of K. Since H_r is irreducible from X to N_{r-1}, it is a subset of L, and since, by Lemma 2, L also contains $M - (K + H_r)$, it contains U. Therefore it follows from the initial supposition that the process described above is an infinite one.

Suppose first that for each i there exists a j such that $N_i \cdot \bar{U}_j = 0$. The

set $X_1 + X_2 \cdots$ has a limit point X. Under the initial supposition there exists a subcontinuum L of M containing X and a point of K but not all of U. If, for each i, L contained $N_i - N_0$, it would contain C. But since $\bar{C} = \bar{U}$, L would then contain U. Hence for some i, L fails to contain $N_i - N_0$. There exists a j such that $N_i \cdot \bar{U}_j = 0$. The point X belongs to $\bar{U}_j$ since all but a finite number of the points $X_1, X_2, \cdots$ belong to U_j and hence, by Lemma 2, L contains $M - (K + \bar{U}_j)$ which contains $N_i - N_0$, a contradiction.

It therefore follows that for some i, $N_i \cdot U_j \neq 0$ for each j. Suppose that the continuum $\bar{U}_i$ is the sum of two proper subcontinua E and F. By Lemma 1(b) one of these, say F, fails to intersect N_i. For some k, $F \cdot N_k \neq 0$. But N_k does not contain E, for if it did, U_k would be a subset of F and hence $N_i \cdot \bar{U}_k = 0$. But then E, $F + N_k$, and $K + N_k$ form a B_2 c. d. of M. It follows from this contradiction that $\bar{U}_i$ is indecomposable. By Lemma 1(c) some composant of $\bar{U}_i$ fails to intersect $K + N_i$. Let X be a point of such a composant and let L be a subcontinuum of M containing X and a point of K. Then L contains $\bar{U}_i$ and, by Lemma 2, also contains $M - (K + \bar{U}_i)$ and hence U, a contradiction of the original assumption.

It follows that there exist points X of U and Y of V such that any subcontinuum of M containing both X and Y contains $U + V$. But, by Lemma 2, such a continuum also contains $M - (\bar{U} + \bar{V})$ and must therefore be M itself. This completes the proof for the case $n = 2$.

Assume now that the sufficiency has been established for $n = k - 1$ and assume that it is not true for $n = k$. Let M be a continuum having property B_k which is not irreducible about any k points. Then it does not have property B_{k-1} since any continuum irreducible about $k - 1$ points is irreducible about k points. There therefore exist continua $H_1, H_2, \cdots, H_k$ which form a B_{k-1} c. d. of M.

By Lemma 1(a) (using the notation of that lemma) each of the sets D_r is connected. It follows from Lemma 1(b) that $\bar{D}_r$ has property B_2, for if K_1, K_2, and K_3 form a B_2 c. d. of $\bar{D}_r$, at least one of them, say K_1, intersects S_r and then $K_1 + K_2$ and $K_1 + K_3$ would form a non-allowable decomposition of $\bar{D}_r$. Hence, by the sufficiency for $n = 2$, $\bar{D}_r$ is irreducible between some two points.

If $\bar{D}_r$ is indecomposable one of its composants fails to intersect S_r by Lemma 1(c). Hence it is irreducible between a point X_r of such a composant and S_r, and every subcontinuum of M containing X_r and a point of S_r contains $\bar{D}_r$. If $\bar{D}_r$ is not indecomposable, let E and F be proper subcontinua whose sum is $\bar{D}_r$. By Lemma 1(b) one of these, say F, fails to intersect S_r. Then $\bar{D}_r$ is irreducible between some point X_r of $\bar{D}_r - E$ and a point of E.

Let L be a subcontinuum of M intersecting both X_r and S_r. Let L' be the closure of the component of $L - L \cdot S_r$ which contains X_r. Because of the irreducibility of $\bar{D}_r$, L' contains $\bar{D}_r - E$. It must also contain E for otherwise the decomposition of $\bar{D}_r$ into the continua E and $L' + \overline{\bar{D}_r - E}$ would contradict Lemma 1(b). In either case, therefore, D_r contains a point X_r such that any subcontinuum of M intersecting both X_r and S_r contains D_r.

By the initial supposition there exists a proper subcontinuum L of M which contains $X_1 + X_2 + \cdots + X_k$. By the preceding paragraph L contains $D_1 + \cdots + D_k$. Since the number of components of $M - L$ is finite, each of them is open in M. Denote one of them by D_{k+1}, and let $S_{k+1} = M - D_{k+1}$. For each r $(r = 1, \cdots, k+1)$, let C_j^r be the closure of the sum of all components of $S_r - (\bar{D}_1 + \cdots + \bar{D}_{r-1} + \bar{D}_{r+1} + \cdots + \bar{D}_{k+1})$ which have limit points in $\bar{D}_j$ $(j = 1, \cdots, r-1, r+1, \cdots, k+1)$. Then $K_j = \bar{D}_j + C_j^1 + \cdots + C_j^{j-1} + C_j^{j+1} + \cdots + C_j^{k+1}$ $(j = 1, \cdots, k+1)$ is a continuum not intersecting $D_1 + \cdots + D_{j-1} + D_{j+1} + \cdots + D_{k+1}$. Moreover $K_1 + \cdots + K_{r-1} + K_{r+1} + \cdots + K_{k+1} = S_r$ $(r = 1, \cdots, k+1)$ and is therefore connected. Hence the continua $K_1, \cdots, K_{k+1}$ form a B_k c. d. of M. This contradiction justifies the induction process and therefore establishes the theorem.

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THE COEFFICIENT OF VISCOUS TRACTION.* [1]

By M. Reiner.

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1. In the first paper the rheological equation of a special case of the generalized Newtonian liquid—the Reynolds Liquid—was derived. While in the generalized Newtonian liquid the tensors of stress and velocity-strain are connected by means of three scalar functions F_0, F_1, F_2 (or $\mathfrak{F}_0, \mathfrak{F}_1, \mathfrak{F}_2$) of *all three* tensor invariants, in the Reynolds liquid these functions are independent of the third invariant and subjected to certain other limitations. The thus specialized functions allow of physical interpretations through "rheological coefficients," viz. the coefficient of dilatancy δ, and the viscosities η_ν and η (or fluidities ϕ_ν and ϕ). We found

$$F_0(0, I'_2, 0)/I'_2 = \delta;\ F_0(e, 0, 0)/3e = \eta_\nu;\ F_1(0, I'_2, 0)/2 = \eta;\ F_2(0, I'_2, 0) = 0 \tag{78}$$ [2]

or

$$\mathfrak{F}_0(0, T'_2, 0) = 0;\ 3\mathfrak{F}_0(T, 0, 0)/T = \phi_\nu;\ 2\mathfrak{F}_1(0, T'_2, 0) = \phi;\ \mathfrak{F}_2(0, T'_2, 0) = 0 \tag{79}$$ [2]

where

$$\phi_\nu = 1/\eta_\nu;\ \phi = 1/\eta \tag{80}$$

from which we see that in certain cases the generalized Newtonian Liquid will behave not differently from the Reynolds Liquid. These cases cover the standard tests of viscometry in the tube- and rotation viscometers from which the rheological equation of the Reynolds liquid was derived in the first paper. In other words: If no other tests were available, the existence of any viscous liquid more general than the Reynolds liquid could not be detected. There exists, however, another basic test with which we did not deal in the first paper and which yields additional information as will be shown.

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[1] This paper forms a continuation of my paper "A mathematical theory of dilatancy" in this Journal, vol. 67 (1945), pp. 350-362, which will be quoted as "the first paper."

[2] By a slip of the pen in the first paper η_ν was written when the intention was to write $3\eta_\nu$ and ϕ_ν was written for $\phi_\nu/3$. This has now been corrected. The present notation makes the analogy of η_ν with the bulk modulus (κ) perfect and is in agreement with the notation used by others.

2. For highly viscous substances to the order of 10^6 to 10^{10} poises such as pitch, shoemaker's wax and others, the standard "shear" methods of viscometry cannot be used. Trouton formed such substances into cylindrical rods which he subjected to pull and observed the rate of elongation. Let T_{zz} be the tractional force per unit area of cross section and e_{zz} the rate of elongation per unit length, then Trouton defined a coefficient of Viscous Traction λ^* by the equation

$$T_{zz} = \lambda^* e_{zz}. \tag{81}$$

The coefficient λ^* is the viscous analogue of Young's modulus and there must be some relation between λ^* and η similar to the relation between Young's modulus and the modulus of rigidity. Trouton attempted to show by reasoning and experiment that $\lambda^*/\eta = 3$, but his not very numerous experiments actually gave an average of $\lambda^*/\eta = 3.16$ and in his reasoning he overlooked that not one of his materials was a *simple* Newtonian liquid. We shall therefore take up the problem as one of the rheological behavior of a *generalized* Newtonian liquid.

3. We have to distinguish in Trouton's experiment two stages. The stress due to the pull T_{zz} has an isotropic component $T_{zz}/3 \cdot \delta_s{}^r$. When the pull is applied, there will be an *initial stage* which starts with an accelerated and ends with a retarded movement of the particles, with between—generally—pulsations. During this initial stage the material expands, the measure of the cubical dilatation at every moment being 3ϵ. This cubical dilatation produces an elastic reaction (hydrostatic tension) $3\kappa\epsilon$. It is accompanied by a viscous resistance $3\eta_v e$ due to volume viscosity. When ϵ has so much increased that $3\kappa\epsilon = T_{zz}/3$, the elastic reaction balances the isotropic component of the pull T_{zz}. The cubical dilatation then ceases to increase and e ultimately vanishes. With this the second stage sets in, in which the movement is steady.

The experiment is arranged in the same way as experiment (iii) of the first paper:—T_{zz} is kept constant and observations start after the steady state of continued elongation at constant rate has been reached. Let λ be the coefficient of viscous traction in this state, then $\lambda = \lambda(0, I'_2, I'_3)$. We are interested in the rheologics of the second stage of steady flow, but it will be profitable to include in the considerations the first stage also.

4. If we neglect the stresses due to the weight of the cylinder and to the inertia of its parts, in short if we neglect in the stress equations all terms in which the density ρ appears as a factor, a homogeneous stress distribution is possible and viscous traction may be defined by

$$T^*{}_s{}^r = T_{zz} \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{82}$$

It will be convenient to introduce an axial unit tensor

$$\gamma_s{}^r = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{83}$$

the first invariant of which is 1/3 and whose deviator is accordingly

$$\gamma'_s{}^r = \gamma_s{}^r - 1/3 \cdot \delta_s{}^r = 1/3 \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{Vmatrix} \tag{84}$$

having the invariants

$$I'_{\gamma 2} = -1/3; \; I'_{\gamma 3} = 2/27. \tag{85}$$

From (84), since $\gamma_a{}^r \gamma_s{}^a = \gamma_s{}^r$,

$$\gamma'_a{}^r \gamma'_s{}^a = 1/3 \cdot \gamma_s{}^r + 1/9 \cdot \delta_s{}^r = 1/3 \cdot \gamma'_s{}^r + 2/3 \cdot \delta_s{}^r. \tag{86}$$

We now write (82)

$$T^*{}_s{}^r = T_{zz} \gamma_s{}^r \tag{87}$$

and have, considering (8),

$$T = T_{zz}/3 - 3\kappa\epsilon; \; T'_s{}^r = T_{zz} \gamma'_s{}^r \tag{88}$$

from which

$$T'_a{}^r T'_s{}^a = T^2{}_{zz}(1/3 \cdot \gamma'_s{}^r + 2/9 \cdot \delta_s{}^r), \; T'_2 = -T^2{}_{zz}/3; \; T'_3 = 2T^3{}_{zz}/27. \tag{89}$$

This gives (compare (49))

$$e'_s{}^r = T_{zz}(\mathfrak{F}_1 + \mathfrak{F}_2 T_{zz}/3) \gamma'_s{}^r. \tag{90}$$

The coefficient of viscous traction as defined by (81) would therefore be

$$(\lambda^*)^{-1} = e/T_{zz} + 2/3 \cdot (\mathfrak{F}_1 + \mathfrak{F}_2 T_{zz}/3) \tag{91}$$

which makes (90)

$$e'_s{}^r = 3/2 \cdot (T_{zz}/\lambda^* - e)\gamma'_s{}^r. \tag{92}$$

In the second stage, when e vanishes, these expressions are reduced to

$$(\lambda)^{-1} = 2/3 \cdot (\mathfrak{F}_1 + \mathfrak{F}_2 T_{zz}/3)\,;\ e'_s{}^r = 3/2 \cdot T_{zz}/\lambda \cdot \gamma'_s{}^r = 3/2 e_{zz}\gamma'_s{}^r. \tag{93}$$

5. Before elaborating these general equations, it will be useful to examine the conditions prevailing in the first approximation of the viscous liquid, namely, the simple Newtonian liquid, the rheological equation of which is

$$T^*_s{}^r = 3(\kappa\epsilon + \eta_v e)\delta_s{}^r + 2\eta e'_s{}^r. \tag{94}$$

From this

$$T^* = 3(\kappa\epsilon + \eta_v e). \tag{95}$$

On the other hand from (82)

$$T^* = T_{zz}/3 \tag{96}$$

and therefore

$$e = (T_{zz} - 9\kappa\epsilon)/9\eta_v. \tag{97}$$

Comparison of (94) with (41) gives

$$F_0(I') = 3\eta_v e\,;\ F_1(I') = 2\eta\,;\ F_2(I') = 0. \tag{98}$$

These expressions are of the same form as (78), but it should be noted that in the present case no restrictions have been placed on the arguments of the functions F nor on the coefficients η and η_v, which need not be constants.

Introducing F_1 and F_2 from (98) into (53), we find

$$\mathfrak{F}_1 = 1/2 \cdot \eta\,;\ \mathfrak{F}_2 = 0 \tag{99}$$

which makes (91)

$$(\lambda^*)^{-1} = 1/3 \cdot \eta + e/T_{zz} \tag{100}$$

and considering (97)

$$\lambda^* = \frac{9\eta\,\dfrac{\eta_v}{1 - 9\kappa\epsilon/T_{zz}}}{\eta + 3\,\dfrac{\eta_v}{1 - 9\kappa\epsilon/T_{zz}}} \tag{101}$$

while, when e vanishes,

$$\lambda = 3\eta. \tag{102}$$

(101) is the viscous analogy to the classical relation between Young's modulus

(E), the modulus of rigidity (μ) and the bulk modulus (κ), viz. $E = 9\mu\kappa/(\mu + 3\kappa)$. It should, however, be noted that the analogy is *not exact*. While $E = 3\mu$ for $\kappa = \infty$ and *only* in this case,[3] the situation is very different with regard to λ^*, as will presently be discussed.

In interpreting (101), one should keep in mind that $1 - 9\kappa\epsilon/T_{zz} \geq 0$, where the sign of inequality is valid in the first, the sign of equality in the second, stage of Trouton's experiment. For the volume viscosity we have $0 \leq \eta_\nu \leq \infty$. It is clear that the volume viscosity cannot be negative as "otherwise the more alternate expansion and compression, alike in all directions, of a fluid, instead of demanding the exertion of work upon it, would cause it to give work out" (Stokes). As can be seen, in the first stage, if η_ν vanishes, λ^* also vanishes. On the other hand $\lambda/\eta = 3$ in both stages if $\eta_\nu = \infty$ and also in the second stage *whatever the magnitude of* η_ν. Because η_ν cannot be negative, λ^*/η cannot exceed the value 3. Therefore $0 \leq \lambda^* \leq 3$, while $\lambda = 3\eta$.

These results are of interest in connection with the rheological equation of the classical viscous liquid or what may be called the Stokes Liquid, for which $T^*{}_s{}^r = -(p + 2\eta e)\delta_s{}^r + 2\eta e_s{}^r$. This was derived by Stokes assuming $\eta_\nu = 0$, but in the first stage of the Trouton experiment a vanishing volume viscosity would mean vanishing viscous resistance against extension of a liquid cylinder, no matter how high the ordinary viscosity η of the liquid—a result at variance with our ideas of viscous flow. Tisza has recently drawn attention to the following quotation from Stokes: . . . "of course we may at once put $\eta_\nu = 0$ if we assume that in the case of a uniform motion of dilatation the pressure at any instant depends only on the actual density and temperature at that instant and not at the rate at which the former changes with the time. In most cases in which it would be interesting to apply the theory of friction of fluids the density of the fluid is either constant or may without sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same and in the third nearly the same whether η_ν were equal to zero or not. Consequently, if theory and experiments should in such cases agree, the experiments must not be regarded as confirming that part of the theory which relates to supposing η_ν to be equal to zero." Examination of (101) shows that Stokes was mistaken in equalizing the influence of either $e = 0$ or $\eta_\nu = 0$ on experimental results. We see that the same result follows from either $e = 0$ or $\eta_\nu = \infty$ and *not* $\eta_\nu = 0$.

It therefore appears that there does not exist a *real* viscous liquid for

[3] Excepting the degenerate case $\mu = 0$.

which the Stokes liquid can serve as a suitable approximation. The first approximation for a viscous liquid is the simple Newtonian liquid possessing *two* constants. This liquid behaves in the more primitive way of the Stokes liquid when $e = 0$ and the Stokes liquid does not represent a *type* of liquid, but rather a rudimentary condition of another type (the simple Newtonian). The Stokes-Navier differential equations should therefore be used only for $e = 0$. The equation $e = 0$ is called the equation of continuity of the incompressible liquid, but it should be noted that while it is true that $e = 0$ is valid for every kind of flow of an incompressible liquid for which $\epsilon = 0$, it also applies to *certain* kinds of flow of *compressible* liquids, e. g., to laminar flow where $\epsilon = 0$, while the liquid is *not incompressible,* and also to such kinds of flow, as in the second stage of Trouton's experiment, where $\epsilon \neq 0$.

6. It is thought that in view of the standard treatment in most textbooks, this question of the legitimacy of the concept of the Stokes liquid and therefore also of the Stokes-Navier differential equations is important enough to warrant some further discussion from another angle. Touton's experiment is the viscous analogy to the Young's modulus experiment dealt with by Murnaghan. We may start, as he does, from the kinematics of the case and assume

$$\dot{u} = qx;\ \dot{v} = qy;\ \dot{w} = rz. \qquad (103)\ ^{4}$$

If we introduce these values into the Stokes-Navier differential equations, we find that the three components of the acceleration a_x, a_y, a_z must vanish or, alternatively, that $\rho a_x, \rho a_y, \rho a_z$ should be negligible. If this is the case, the stress will be homogeneous, so that the assumption (103) corresponds with the assumption (82).

From (103) we find

$$e_s{}^r = \begin{Vmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & r \end{Vmatrix};\ e = (2q + r)/3;\ e'_s{}^r = (r - q)\gamma'_s{}^r. \qquad (104)$$

For the simple Newtonian liquid

$$T^*_s{}^r = [-p + \eta\nu(2q + r)]\delta_s{}^r + 2\eta(r - q)\gamma'_s{}^r \qquad (105)$$

The numbers q and r must be such that the expressions on the right side of (105) and (82) can be equated.

This requires

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[4] I have replaced Murnaghan's p by q, having disposed of p before.

$$q = (3p + T_{zz})/9\eta_v - T_{zz}/6\eta; \quad r = (3p + T_{zz})/9\eta_v + T_{zz}/3\eta$$
$$e = (3p + T_{zz})/9\eta_v; \quad e' = T_{zz}/2\eta \cdot \gamma'_s{}^r; \tag{106}$$

$$q/r = [2\eta(3p + T_{zz}) - 3\eta_v T_{zz}] / [2\eta(3p + T_{zz}) + 6\eta_v T_{zz}]. \tag{107}$$

The quotient q/r is the viscous analogy to Poisson's ratio. If we assume vanishing η_v, we find $q = \infty$, $r = \infty$, $e = \infty$, $q/r = 1$. The expansion of the material in the first stage would be purely elastic, instantaneous and be completed in no time. This is what would be expected in the absence of the damping influence of a volume viscosity. If, on the other hand, η_v would be so great that it may be put $= \infty$, we find $q = -T_{zz}/6\eta$; $r = T_{zz}/3\eta$; $e = 0$; $q/r = -\frac{1}{2}$ and the expansion would be infinitely delayed so that the second stage would never be reached. It may be mentioned that Tisza deduces from observations on supersonic absorption in certain liquids for η_v/η a value of 2000. This, for the purpose of our calculations, is practically infinite.

Applying (104) to the Stokes liquid, we get

$$p = -2\eta(r - q)/3; \; p + T_{zz} = 4\eta(r - q)/3 \tag{108}$$

from which

$$q = r - T_{zz}/2\eta; \; e = r - T_{zz}/3\eta; \; q/r = 1 - T_{zz}/3r\eta; \; \lambda^* = T_{zz}/r. \tag{109}$$

If inertia terms are neglected, there is no way of determining r and the problem is indeterminate. It becomes determinate if e vanishes, which gives $r = T_{zz}/3\eta$, $q/r = 0$ and $\lambda = 3\eta$. If the meaning of Trouton's experimental results is that in a viscous liquid in a first approximation $\lambda/\eta = 3$, this shows that Stokes' relation between stress and strain makes sense only if e is assumed to vanish. This does not mean that the liquid must be incompressible, but only that it is considered in a state when no change of volume takes place or after all such change has taken place.

7. We may now take up the general problem where we left it at the end of **4**. We shall, however, in the following deal with the second stage only, when e vanishes.

In viscous traction, while stresses may be high, the rate of strain is small. As a matter of fact this test was devised for such materials where large stresses produce small velocity strains only. In order to make use of this fact, we have to express λ in terms of *strain*.

For this we use (41), which gives, considering the second of equations (88), the second of equations (93) and (86) and (85),

$$T_{zz}\gamma'_s{}^r = F_1(I') \cdot 3/2 \cdot T_{zz}/\lambda \cdot \gamma'_s{}^r + F_2(I') \cdot 3/4 \cdot T^2{}_{zz}/\lambda^2 \cdot \gamma'_s{}^r \tag{110}$$

This yields

$$\lambda = 3/2 \cdot [F_1(0, I'_2, I'_3) + \tfrac{1}{2}F_2(0, I'_2, I'_3)e_{zz}]. \qquad (111)$$

Considering (78), we now develop the functions F_1 and F_2 into power series of the argument I'_3 as follows

$$F_1(0, I'_2, I'_3) = 2(\eta + \eta_1 I'_3 + \eta_2 I'^2_3 + \cdots); \; F_2(0, I'_2, I'_3) = cI'_3 + c_1 I'^2_3 + \cdots \qquad (112)$$

so that generally

$$T'^r_s = 2(\eta + \eta_1 I'_3 + \cdots)e'^r_s + (cI'_3 + \cdots)(e'^r_\alpha e'^\alpha_s + 2/3 \cdot I'_2 \cdot \delta^r_s) \; . \qquad (113)$$

The coefficients η and c will generally be functions of the velocity strain components through the invariant I'_2. However, if we assume that these are all of the same order, or if they are constants, we may neglect the terms in I'^2_3 and higher powers and postulate a liquid which we may call the Trouton Liquid, with a rheological equation accordingly.

Its coefficient of viscous traction is from (111) and (112)

$$\lambda = 3\eta(1 + \eta_1/\eta \cdot I'_3 + c/4\eta \cdot I'_3 \cdot e_{zz}) \qquad (114)$$

or introducing $I'_3 = e_{zz}^3/4$,

$$\lambda/\eta = 3(1 + 1/4 \cdot \eta_1/\eta \cdot e_{zz}^3 + 1/16 \cdot c/\eta \cdot e_{zz}^4) \; . \qquad (115)$$

We have seen that for the simple Newtonian liquid, even should the coefficients of viscosity be not constant, λ/η cannot surpass the value 3. This is also the case with regard to the Reynolds liquid. Should, therefore, the average value of 3.16 found by Trouton be of significance, which it is difficult to judge, this would indicate the presence of the rheological coefficients η_1 or c or both. The term to which η_1 is attached changes its sign with the direction of the traction. In contradistinction the term to which c is attached does not do so. Experiments of viscous traction in *push* instead of pull, which Trouton did *not* carry out, would accordingly furnish criteria for the retention of one or the other of the two coefficients peculiar to the Trouton liquid.

8. Conclusions. The rheological equation of the most general Newtonian liquid so far suggested by experimental evidence is

$$T^{*r}_s = (3\kappa\epsilon + 3\eta_\nu e + \delta I'^r_s)\delta^r_s + 2(\eta + \eta_1 I'_3)e'^r_s + cI'_3(e'^r_\alpha e'^\alpha_s + 2/3 \cdot I'_2\delta^r_s) \qquad \text{(i)}$$

which may be called the Trouton Liquid—where T^{*r}_s is the total stress (elastic and viscous), 3ϵ the cubical dilatation, $3e$ the rate thereof, e'^r_s the deviator of the velocity strain and κ, η_ν, δ, η, η_1, c are six rheological coeffi-

cients. If a cylinder of the material is subjected to a simple axial traction T_{zz} and extends with the rate e_{zz}, the coefficient of viscous traction after a steady state has been reached, i. e., when $e = 0$, is

$$\lambda = T_{zz}/e_{zz} = 3\eta(1 + \eta_1/4\eta \cdot e_{zz}^3 + c/16\eta \cdot e_{zz}^4) \ . \qquad \text{(ii)}.$$

When η_1 and c vanish, the liquid is a Reynolds Liquid. The Trouton Liquid behaves in the same way as the Reynolds Liquid in tests for which I'_3 vanishes, e. g. in laminar flow. When, in addition, δ vanishes the liquid is a simple Newtonian Liquid. For cases of flow in which $e = 0$, the simple Newtonian Liquid is reduced to the Stokes Liquid. The rheological equation $T^*{}_s{}^r = -(p + 2\eta e)\delta_s{}^r + 2\eta e_s{}^r$ from which the Stokes-Navier differential equations in their general form are derived leads in viscous traction to unacceptable results. The Stokes-Navier differential equations should therefore be used in conjunction with $e = 0$ only.

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For bibliography see first paper.

AN ARITHMETIC CHARACTERIZATION OF PROPER CHARACTERISTICS OF LINEAR SYSTEMS.*

By G. B. Huff.

Introduction. A *linear system* Σ of algebraic curves in a plane Π may be given by an equation of the form

$$\lambda_0 f_0 + \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_d f_d = 0, \tag{1}$$

where $f_0, f_1, f_2, \cdots, f_d$ are $d+1$ linearly independent ternary polynomials of *order* x_0, and $\lambda_0, \lambda_1, \cdots, \lambda_d$ are parameters; d is said to be the *dimension* of the system. This article is concerned with linear systems which are defined by giving the order x_0 and the *multiplicities* $x_1, x_2, \cdots, x_\rho$ at a set of ρ general points $P_1, P_2, \cdots, P_\rho$ called the *base* of the system. If a system contains all the curves of order x_0 with the stipulated multiplicities at the base, it is said to be *complete*. If the conditions imposed by the base are independent, the system is said to be *regular*. Suppose, further, that at least one curve of the system is irreducible, has no multiple points outside the base, and is of genus p. For such a complete and regular linear system, its characteristic $x \equiv \{x_0; x_1, x_2, \cdots, x_\rho\}$ satisfies Cremona's equations [2]:

$$\begin{aligned} x_0^2 - x_1^2 - x_2^2 - \cdots - x_\rho^2 &= d + p - 1 \\ 3x_0 - x_1 - x_2 - \cdots - x_\rho &= d - p + 1. \end{aligned} \tag{2}$$

Since there is an irreducible curve in the system, a line through P_1, P_2 may not meet this curve in more than x_0 points; a conic through $P_1, P_2, \cdots, P_5$ may not meet the curve in more than $2x_0$ points, etc. Thus the characteristic of a linear system satisfies the inequalities [3]:

$$\begin{aligned} x_0 - x_1 - x_2 &\geq 0 \\ 2x_0 - x_1 - x_2 - \cdots - x_5 &\geq 0 \\ 3x_0 - x_1 - x_2 - \cdots - x_9 &\geq 0 \\ 3x_0 - 2x_1 - x_2 - \cdots - x_7 &\geq 0 \\ \cdots\cdots\cdots\cdots\cdots & \end{aligned} \tag{3}$$

* Received April 3, 1946. Presented to the American Mathematical Society April 26, 1946.

It is understood that this system of inequalities includes all those of the form:

$$(3') \qquad y_0x_0 - y_1x_1 - y_2x_2 - \cdots - y_\rho x_\rho \geq 0\,,$$

where $y \equiv \{y_0; y_1, \cdots, y_\rho\}$ is the characteristic of a curve (linear system with $d = 0$) of lower order than x_0. If these inequalities hold for x so ordered that $x_1 \geq x_2 \geq \cdots \geq x_\rho$ and with all the characteristics y similarly ordered, it is clear that they hold *a fortiori* for all other orderings.

In 1934, Coble [2] gave a determination of all the integer solutions of Cremona's equations (2) for any given values of p, d, ρ. He used the word characteristic for any solution of (2) and characteristics were classified as *proper, degenerate* or *virtual* according as the general curve of the system is existent and irreducible, existent but reducible, or non-existent.

These results underlined the need for arithmetic criteria for proper characteristics. In this paper that problem is investigated and it is in fact shown that, *for early values of p, d, every set of integers satisfying* (2) *and* (3') *is the characteristic of a linear system.*

A basic tool is the notion of the image of a characteristic x under A_{123}. Let a quadratic transformation be set up between the plane Π and a plane Π', with fundamental points at P_1, P_2, P_3 in Π and call the fundamental points in Π', P_1', P_2', P_3'. Designate the images of $P_4, P_5, \cdots, P_\rho$ in Π' by P_4', $P_5', \cdots, P_\rho'$. The image of Σ is a linear system Σ' in Π', determined by a characteristic x' at the base $P_1', P_2', P_3', \cdots, P_\rho'$; and x' is given by

$$(4) \qquad A_{123}: \quad \begin{aligned} x_i' &= x_i + (x_0 - x_1 - x_2 - x_3) && i = 0, 1, 2, 3 \\ x_j' &= x_j && j = 4, 5, \cdots, \rho. \end{aligned}$$

Given a proper characteristic, the linear transformation A_{123} may be used to construct a new proper characteristic; moreover, if x' is proper, it follows that x is proper.

Since the points of a base may be numbered at will, the image of a characteristic x under any permutation of $x_1, x_2, \cdots, x_\rho$ is proper if and only if x is. Thus A_{123} and the interchanges (x_1x_2), $(x_1x_3), \cdots, (x_1x_\rho)$ generate a linear transformation group, $\mathfrak{G}_\rho$, with the property that if x is a proper characteristic and $S \in \mathfrak{G}_\rho$, then $x' = S(x)$ is proper. The bilinear form $y_0x_0 - y_1x_1 - \cdots - y_\rho x_\rho$ of the inequalities (3') is invariant under A_{123} and hence for any $S \in \mathfrak{G}_\rho$.

The line L through P_1, P_2, a linear system of $p = d = 0$ and charac-

teristic $l = \{1; 1, 1, 0, \cdots, 0\}$ is a principal curve for the quadratic transformation with fundamental points at P_1, P_2, P_3. Its image under this transformation is the set of directions about P_3'. The image of l under A_{123} is $\{0; 0, 0, -1, 0, \cdots, 0\}$. This special characteristic is defined to be proper and to represent the directions about P_3'.

1. The inequality $x_0 - x_1 - x_2 - x_3 \geq 0$. Two characteristics x, x' are said to be of the same *type* if x' is the image of x under a permutation of $x_1, x_2, \cdots, x_\rho$. If x has the property $x_0' - x_1' - x_2' - x_3' \geqq 0$ for all x' of the same type, it is clear that its image under A_{123} is of the same or higher order, and that the same would be true for any x' of the same type. Such a characteristic is said to be of *minimal order*. Noether [7], Bertini [1], Guccia [4], and others have obtained a variety of results along these lines. Since in each of these papers the characteristics were all assumed to be proper, the theorems as stated are not directly applicable here. The arithmetical results were obtained on the assumption that $x_0, x_1, \cdots, x_\rho$ were non-negative. If we call such a characteristic non-negative, the results may be stated as follows:

(5) (Noether) *The only non-negative characteristic of minimal order for $p = 0$, $d = 2$ is* $\{1; 0, 0, \cdots, 0\}$.

(6) (Noether) *The only non-negative characteristics of minimal order for $p = 0$, $d = 1$ are of type* $\{1; 1, 0, \cdots, 0\}$.

(7) (Bertini) *There is no non-negctive characteristic of minimal order for $p = d = 0$, but those of type* $\{0; 0, \cdots, 0, -1\}$ *are of minimal order.*

(8) (Bertini-Guccia) *The only non-negative characteristics of minimal order for $p = 1$, $d = 0$ are of type* $\{3\mu; \mu^9\, 0^{\rho-9}\}$.

(9) (Guccia) *The only non-negative characteristics of minimal order for $p = 1$, $d = 1, 2, \cdots, 7$ are of type* $\{3; 1^{9-d}, 0, \cdots, 0\}$.

These are readily proved without reference to the original papers by the following device, (which is essentially what all these early workers used): in (2) set $x_0 = x_1 + x_2 + x_3 + \delta$ and then subtract x_3 times the second from the first, getting:

$$(*)\quad \delta^2 + (2x_1 + 2x_2 - x_3)\delta + 2(x_1x_2 - x_3^2) + x_4(x_3 - x_4) + \cdots + x_\rho(x_3 - x_\rho) = (d + p - 1) - x_3(d - p + 1).$$

Now if x is non-negative and ordered so that $x_1 \geq x_2 \geq \cdots \geq x^\rho$, the results (5), $\cdots$, (9) are easily established by substituting the values of p, d in (*) and making quite elementary arguments.

From the arithmetic proposition (7) Bertini obtained a geometric theorem which will be stated as follows for use in the next section:

(10) *If l is a proper solution of* (2) *for $p = d = 0$, there exists an $S \in \mathfrak{G}_\rho$ (i. e., an S which is the product of transformations A_{123} and permutations) such that $S(l) = \{0; 0, 0, \cdots, 0, -1\}$.*

The curve defined by a proper l is a rational curve determined by its behavior at a base, a Bertini L-curve. The totality of L-curves for a given base is the same as the collection of all principal curves of homaloidal nets defined at that base.

2. Property *A* and arithmetic inequalities. In a recent article [6] this writer studied characteristics x with $x_0 > 0$ and p, d non-negative. This was designated as property A. Of the results obtained there, the following ones are needed.

(11) *If x^* is obtained from x by setting $x_\rho = 0$, then*

$$p^* = p + x_\rho(x_\rho - 1)/2, \qquad d^* = p + x_\rho(x_\rho + 1)/2.$$

If x has property A, then so does x^*.

(12) *If x has property A and c is the characteristic of a homaloidal net, then $c_0x_0 - c_1x_1 - c_2x_2 - \cdots - c_\rho x_\rho \geq 0$. Moreover, the equality sign holds only for the characteristics of the principal curves of the homaloidal net.*

These will be used in establishing the theorem below, which is a substantial generalization of Theorem 3 of the earlier paper [6].

(13) *If x has property A and l is a proper characteristic of $p = d = 0$ and $l_0x_0 - l_1x_1 - l_2x_2 - \cdots - l_\rho x_\rho < 0$, then $x_0 \geq l_0$. Moreover, $x_0 = l_0$ only if $x = l$.*

If x is a proper characteristic of $p = d = 0$, the case is quickly settled.

x and l both represent irreducible Bertini L-curves which meet at more than $x_0 l_0$ points and coincide. Thus $x = l$.

If x is not the characteristic of a Bertini L-curve, the reasoning goes as follows. Since l is proper, according to (10) there exists an $S \in \mathfrak{S}_\rho$ such that $\bar{l} = S(l) = \{0; 0, \cdots, 0, -1\}$. From the fact that x is not the characteristic of a Bertini L-curve, it is clear from (12) that $\bar{x} = S(x)$ has $\bar{x}_0 > 0$. The invariance of the bilinear form $y_0x_0 - y_1x_1 - y_2x_2 - \cdots - y_\rho x_\rho$ under S implies that

$$l_0x_0 - l_1x_1 - \cdots - l_\rho x_\rho = \bar{l}_0\bar{x}_0 - \bar{l}_1\bar{x}_1 - \cdots - \bar{l}_\rho\bar{x}_\rho = \bar{x}_\rho,$$

and $\bar{x}_\rho$ is negative, say $\bar{x}_\rho = -k$, $k > 0$. Thus $\bar{x}$ may be written in the form

$$\bar{x} = \bar{x}^* + k\bar{l},$$

where $\bar{x}^*$ is $\bar{x}$ with $\bar{x}_\rho$ replaced by zero and k is a positive integer. Consider now the image of $\bar{x}$ under S^{-1}.

$$S^{-1}(\bar{x}) = S^{-1}(\bar{x}^* + k\bar{l}) = S^{-1}(\bar{x}^*) + kS^{-1}(\bar{l}),$$

or

$$x = (\bar{x}^*)' + kl.$$

By (11), $\bar{x}^*$ has $p^* + d^* = p + d + k^2 > 0$ and by (12), $(\bar{x}_0^*)' > 0$. From

$$x_0 = (\bar{x}_0^*)' + kl_0,$$

it follows that

$$x_0 > l_0.$$

This result is put into the following form to make it more usable in the next proof.

(14) *If l is a proper characteristic of $p = d = 0$, and x is any other characteristic of property A and $x_0 \leq l_0$, then $l_0x_0 - l_1x_1 - l_2x_2 - \cdots - l_\rho x_\rho \geq 0$.*

Any characteristic that satisfies the inequalities (3′) and has $x_1, x_2, \cdots, x_\rho$ non-negative satisfies $l_0x_0 - l_1x_1 - l_2x_2 - \cdots - l_\rho x_\rho \geq 0$ for all proper characteristics l of $p = d = 0$ and $l_0 < x_0$, including in particular the special

proper characteristics of type $\{0; 0, 0, \cdots, 0, -1\}$. For such an x the following statement may be made.*

THEOREM I. *If x has property A and if*

$$l_0x_0 - l_1x_1 - l_2x_2 - \cdots - l_\rho x_\rho \geq 0$$

for all proper l of $p = d = 0$ and $l_0 < x_0$; and x' is the image of x under any $S \in \mathfrak{G}_\rho$, then either $x_0' > 0$ and $x_i \geq 0$; or x' is of type $\{0; 0, 0, \cdots, 0, -1\}$

Suppose that x is not a proper characteristic of $p = d = 0$. The hypothesis of the theorem and (14) assure us that $l_0x_0 - l_1x_1 - l_2x_2 - \cdots - l_\rho x_\rho \geq 0$ for all proper l. Now the set $\{l\}$ is merely permuted by any $S \in \mathfrak{G}_\rho$ and the bilinear relation $l_0x_0 - l_1x_1 - \cdots - l_\rho x_\rho$ is invariant under S. Thus $l_0x_0' - l_1x_1' - \cdots - l_\rho x_\rho' \geq 0$ for all proper l. Since these include those of type $\{0; 0, 0, \cdots, 0, -1\}$, it follows that $x_i \geq 0$. By (12), $x_0' > 0$.

If x is indeed a proper characteristic of $p = d = 0$, its image under any $S \in \mathfrak{G}_\rho$ is proper and must have $x_0' > 0$, $x_i' \geq 0$ or must be of type $\{0; 0, 0, \cdots, 0, -1\}$.

3. Arithmetic criteria for proper characteristics.

In this section it will be shown that simple arithmetic conditions exist which assure the properness of characteristics for eleven values of p, d. Two of the cases are somewhat different from the others and are treated separately.

THEOREM II. *If x is a non-negative characteristic of $p = d = 0$; such that $l_0x_0 - l_1x_1 - \cdots - l_\rho x_\rho \geq 0$ for all proper characteristics l of $p = d = 0$ and $l_0 < x_0$, then x is proper.*

By Theorem (7), x is not of minimal order. Let p be the permutation that sends the three highest multiplicities into x_1, x_2, x_3 and let $x' = A_{123}p(x)$. It is clear that $x_0 > x_0'$. By Theorem I, either x_0' is of type $\{0; 0, 0, \cdots, 0, -1\}$ or it is non-negative. In the first case x' is proper and in the second x' is not of minimal order and the process may be continued. Since $x_0 > x_0' > x_0'' > \cdots$ is a sequence of non-negative integers, there must finally result some image x^* of type $\{0; 0, 0, \cdots, 0, -1\}$. Thus x is proper.

An entirely similar argument yields:

---

* The author is indebted to R. J. Walker for the conjecture that this theorem might be valid.

THEOREM III. *If x is a non-negative characteristic of p, $d = 1, 0$ such that the g. c. d. of $x_0, x_1, \cdots, x_\rho$ is 1 and $l_0x_0 - l_1x_1 - l_2x_2 - \cdots - l_\rho x_\rho \geq 0$ for all proper l of $p = d = 0$ and $l_0 < x_0$, then x is proper.*

Now by Theorem (8), x is of type $\{3\mu; \mu^9, 0, \cdots, 0\}$ or is not of minimal order. In the first case $\mu = 1$ and x is proper. As before let p be a permutation such that $p(x)$ does not satisfy $x_0 - x_1 - x_2 - x_3 \geq 0$ and consider $x' = A_{123}p(x)$. Again $x_0 > x_0'$ and either $x' = \{3\mu; \mu^9, 0, \cdots, 0\}$ or it is not of minimal order. Thus a sequence of characteristics $x, x', \cdots, x^*$ can be set up so that $x_0 > x_0' > \cdots > x_0^*$, where $x^* = \{3\mu; \mu^9, 0, \cdots, 0\}$. But the g. c. d. of $x_0^*, x_1^*, \cdots, x_\rho^*$ is 1, and hence $x^* = \{3; 1^9, 0, \cdots, 0\}$ and x is proper.

These arguments indicate clearly the method of proof to be used in establishing:

THEOREM IV. *If x is a characteristic of $p, d = 0, 1; 0, 2; 1, d$ $(d = 1, 2, \cdots, 7)$ such that $l_0x_0 - l_1x_1 - \cdots - l_\rho x_\rho \geq 0$ for all proper l of $p = d = 0$ and $l_0 < x_0$, then x is proper.*

These theorems are all actually necessary and sufficient condition theorems, since proper positive ordered characteristics of these sorts must satisfy the inequalities (3). However, the interesting thing is the sufficiency part of the theorems.

Conclusion. The question raised in the introduction must then be answered in the affirmative for the values of p, d considered in Section **3**. Any characteristic x which satisfies (2) and (3′) for those values of p, d must be proper. Indeed, it is sufficient that x satisfy (3′) for the characteristics of Bertini L-curves. This result was already known [5] for $p = 0$, $d = 2$, but was, in a sense, vacuous since there existed no arithmetic criterion for $p = d = 0$. Thus the validity of Theorem II adds to the significance of the other results.

To construct arithmetically a table of proper characteristics of Bertini L-curves $(p = d = 0)$, classified according to order x_0, one might proceed as follows. Suppose that the table has been computed up to $x_0 = n - 1$. Then for the order n, there is a finite number of solutions of (2) in non-negative integers. All these could be tested in terms of the earlier ones. The process could be carried as far as desired with (a) certainty of getting all proper

characteristics of each order and (b) no danger of including improper characteristics. This table could then be used to check construction of tables for other values of p, d.

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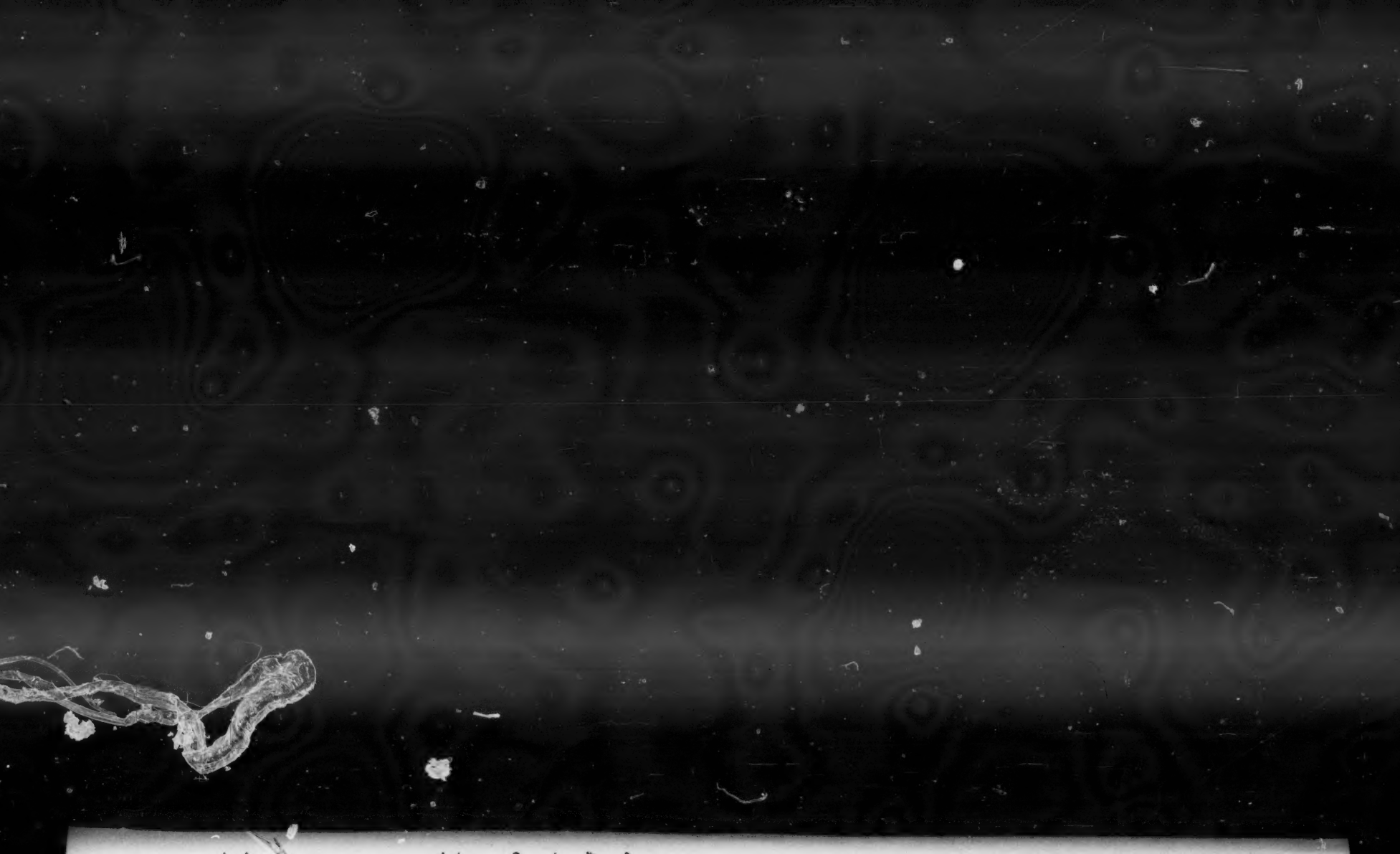

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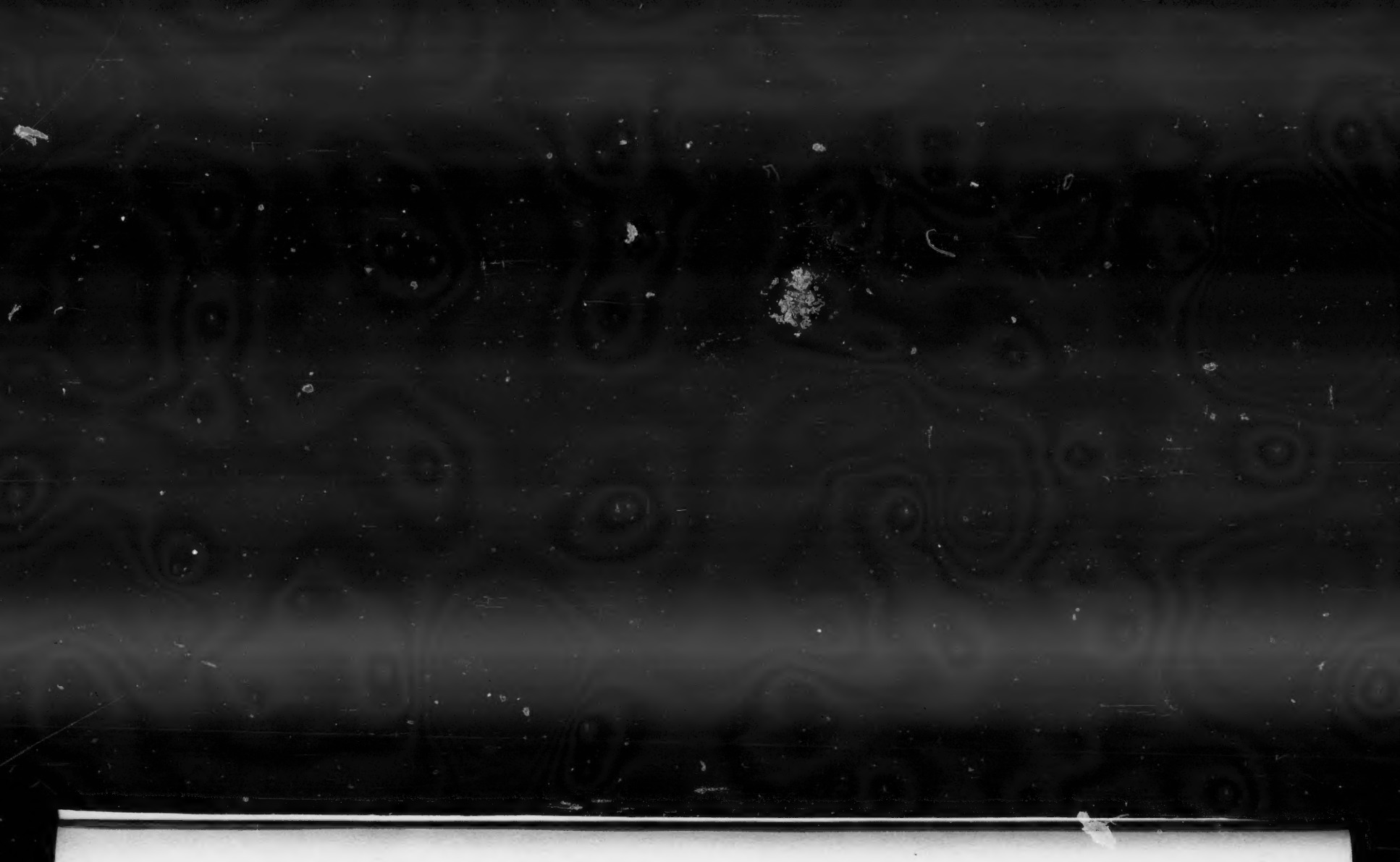
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